Week 8, lecture 1:
Matrix form of a linear system. Inverses.
MA180/185/190 Algebra

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Reminder. Assignment 3 is due on Friday 8 Nov at 3pm.

Matrix Algebra

Matrix form of a system of linear equations

Properties. Inverses.

Matrix form of a system of linear equations

Matrices with only one row or only one column are also called **vectors**. For instance, the following is a 3×1 matrix (or also a "column vector"):

$$\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

As for matrices of other sizes, vectors can be added and multiplied (provided that the sizes allow for that). For example, we can compute the following product between a 3×3 matrix and a column 3×1 vector:

$$A\mathbf{v} = \begin{pmatrix} 2 & 0 & 3 \\ 1 & -1 & 2 \\ 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 11 \\ 5 \\ 11 \end{pmatrix}$$

Matrix form of a system of linear equations

We can therefore apply matrices (and matrix product) to systems of linear equations. Consider the following system from one of our previous examples:

$$\begin{cases} 2x + 3y + z = 7 \\ 2x + y + 3z = 9 \\ 4x + 2y + 5z = 16 \end{cases}$$

We can rewrite this system as an equation involving the following matrix and vectors. Let $A = \begin{pmatrix} 2 & 3 & 1 \\ 2 & 1 & 3 \\ 4 & 2 & 5 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 7 \\ 9 \\ 16 \end{pmatrix}$.

Then solving our system is equivalent to solving the equation

$$A\mathbf{x} = \mathbf{b}.$$

Idea: "invert" A and find $\underline{x}$ as

$\underline{x} = A^{-1}\underline{b}$

Arithmetic properties

Provided they can be performed, matrix operations satisfy the following properties:

- $A + B = B + A$
- $A + (B + C) = (A + B) + C$
- $A(BC) = (AB)C$
- $A(B + C) = AB + AC$
- $(B + C)A = BA + CA$

Note. We have seen that the order in which we multiply matrices matters, so we need to make sure to multiply them in the correct order!

Zero matrix

A matrix whose entries are all zero is called a **zero matrix**. For instance, the following

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$

are zero matrices of various sizes.

Denote by $O_{m\times n}$ the matrix with m rows and n col's of 0s, and if A is $m \times n$ then

$$A + O_{m\times n} = O_{m\times n} + A = A$$

Diagonal matrices

Diagonal matrices are square matrices whose only non-zero entries are those on the main diagonal (i.e. for a 3×3 matrix A these are the entries a_{11}, a_{22} and a_{33}). For instance,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

are a 3×3 and a 2×2 diagonal matrix, respectively.

Identity matrices

The $n \times n$ **identity matrix** is the $n \times n$ diagonal matrix whose diagonal entries are all 1. We write $I_{n\times n}$ for the $n \times n$ identity matrix (or just I if the size is clear from the context).

For instance,

$$I_{1\times 1} = (1), \quad I_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad I_{3\times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Important property: for any $n \times n$ matrix A,

$$A \cdot I_{n\times n} = I_{n\times n} \cdot A = A$$

Inverses

Inverse matrix

If A is a square matrix, and if there exists a matrix B of the same size for which $AB = BA = I$, then A is said to be **invertible** (or nonsingular) and B is called an inverse of A, denoted A^{-1}. If no such matrix B exists, then A is said to be singular.

Question. How do we compute, if it exists, the inverse of a matrix?

Hint In your assignment, you are given two matrices whose product is a scalar multiple of the identity. That tells you that these matrices are very close to being the inverse of one another.

Inverse of a 2×2 matrix

The following simple rule applies in the case of a 2×2 matrix.

Theorem

The matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible if and only if the number $\underbrace{ad - bc}_{\text{also called the determinant of } A}$ is non-zero. In this case, the inverse is given by the formula

$$A^{-1} = \frac{1}{\underbrace{ad - bc}_{\text{scalar multiplication}}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

We can easily verify that A^{-1} is the inverse of A.

Example

Example. Compute, if it exists, the inverse of $A = \begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix}$.

$\to$ Invertible? $2\cdot(-1) - 1\cdot(-3) = -2+3 = 1 \neq 0$ so $\checkmark$ invertible

$\to$ Apply formula for A^{-1}:

$$A^{-1} = \frac{1}{1}\begin{pmatrix} -1 & -1 \\ 3 & 2 \end{pmatrix}$$

$\Big($ $\to$ Check: $A\cdot A^{-1} = \begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_{2\times 2}$ $\checkmark$ $\Big)$

Gaussian elimination for inverses

In general we can apply the following variant of Gaussian elimination to determine, if it exists, the inverse of a square matrix A.

Suppose we are given the following 3×3 matrix:

$$A = \begin{pmatrix} 2 & 0 & 4 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}.$$

To find its inverse, we form a new type of augmented matrix, by putting a 3×3 identity matrix next to it:

$$\left(\begin{array}{ccc:ccc} 2 & 0 & 4 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right).$$

Gaussian elimination for inverses

We then apply the usual elementary row operations with the following goal: bring (if possible) the augmented matrix to the form:

$$\left(\begin{array}{ccc:ccc} 1 & 0 & 0 & * & * & * \\ 0 & 1 & 0 & * & * & * \\ 0 & 0 & 1 & * & * & * \end{array}\right)$$

(where the entries on the right side of the dashed line will be determined by the row operations we apply). If (!) we can get to this form, then the matrix appearing on the right side of the dashed line is the inverse of the initial matrix A!

Let's see if our matrix A has an inverse.

Gaussian elimination for inverses

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 4 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 - R_1} \left(\begin{array}{ccc|ccc} 2 & 0 & 4 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -3 & -1 & 0 & 1 \end{array}\right) \xrightarrow{\frac{1}{2}R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1/2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -3 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 - R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1/2 & 0 & 0 \\ 0 & 1 & -2 & -1/2 & 1 & 0 \\ 0 & 1 & -3 & -1 & 0 & 1 \end{array}\right) \xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1/2 & 0 & 0 \\ 0 & 1 & -2 & -1/2 & 1 & 0 \\ 0 & 0 & -1 & -1/2 & -1 & 1 \end{array}\right) \xrightarrow[R_2 - 2R_3]{R_3 \cdot (-1)}$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 3 & -2 \\ 0 & 0 & 1 & 1/2 & 1 & -1 \end{array}\right) \xrightarrow{R_1 - 2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & -2 & 2 \\ 0 & 1 & 0 & 1/2 & 3 & -2 \\ 0 & 0 & 1 & 1/2 & 1 & -1 \end{array}\right)$$

So A^{-1} exist, and $A^{-1} = \begin{pmatrix} -1/2 & -2 & 2 \\ 1/2 & 3 & -2 \\ 1/2 & 1 & -1 \end{pmatrix}$

Gaussian elimination for inverses

Tip. Once done, you can verify your work by computing the product of the initial matrix and the inverse found through Gaussian elimination. You should get the identity matrix!

Exercise. You can apply this new method to compute the inverse of
$$\begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix}.$$

We have computed it before, so you should get the same result.

Soln:
$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ -3 & -1 & 0 & 1 \end{array}\right) \xrightarrow{R_2 + \frac{3}{2}R_1} \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & \frac{-1+3}{2} & \frac{3}{2} & 1 \end{array}\right) = \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & 1 \end{array}\right) \xrightarrow{2 \cdot R_2}$$

$$\to \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & 1 & 3 & 2 \end{array}\right) \xrightarrow{R_1 - R_2} \left(\begin{array}{cc|cc} 2 & 0 & -2 & -2 \\ 0 & 1 & 3 & 2 \end{array}\right) \xrightarrow{R_1 \cdot \frac{1}{2}} \left(\begin{array}{cc|c} 1 & 0 & \boxed{\begin{matrix} -1 & -1 \\ 3 & 2 \end{matrix}} \\ 0 & 1 & \end{array}\right)$$

This is our inverse!

Example

Example. Consider the following system of linear equations:

$$\begin{cases} 5x + 3y + 2z = 4 \\ 3x + 3y + 2z = 2 \\ y + z = 5 \end{cases}$$

1. Write the system in matrix form.
2. Solve the system by inverting the coefficient matrix.

1. $$\begin{pmatrix} 5 & 3 & 2 \\ 3 & 3 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}$$

Example

2. We find the inverse of A by applying Gaussian elimination to the augmented matrix

we'll start from here tomorrow!

$$\left(\begin{array}{ccc|ccc} 5 & 3 & 2 & 1 & 0 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

Example

Our solution is then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{3}{2} & \frac{5}{2} & 2 \\ \frac{3}{2} & -\frac{5}{2} & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix} =$$