Coloured shuffle compatibility, Hadamard products, and ask zeta functions

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We devise an explicit method for computing combinatorial formulae for Hadamard products of certain rational generating functions. The latter arise naturally when studying so-called ask zeta functions of direct sums of modules of matrices or class- and orbit-counting zeta functions of direct products of nilpotent groups. Our method relies on shuffle compatibility of coloured permutation statistics and coloured quasisymmetric functions, extending recent work of Gessel and Zhuang.

1 Introduction

Permutation statistics and shuffle compatibility. Permutation statistics are functions defined on permutations and their generalisations. Studying the behaviour of said functions on sets of permutations is a classical theme in algebraic and enumerative combinatorics. The origins of permutation statistics can be traced back to work of Euler and MacMahon. The past decades saw a flurry of further developments in the area; see e.g. [4–6, 19, 28, 30, 38, 41].

Recently, Gessel and Zhuang [20] developed an algebraic framework for systematically studying what they dubbed *shuffle-compatible* permutation statistics by means of associated shuffle algebras. In their work, *quasisymmetric functions* (and the closely related P-partitions) as well as *Hadamard products* of rational generating functions played key roles. For further developments building upon [20], see e.g. [21].

Zeta functions and enumerative algebra. Numerous types of *zeta functions* have been employed in the study of enumerative problems surrounding algebraic structures. L. Solomon [37] introduced zeta functions associated with integral representations. In another influential paper, Grunewald, Segal, and Smith [22] initiated the study of zeta functions associated with (nilpotent

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2020 Mathematics Subject Classification. 05A15, 11M41, 20D15, 20E45, 05E05, 15B33
Keywords. Coloured permutations, permutation statistics, Hadamard products, shuffle compatibility, average sizes of kernels, zeta functions

and pro-p) groups. Among the counting problems considered in [22] are the enumeration of subgroups and normal subgroups of suitable groups according to their indices. The associated zeta functions are then Dirichlet generating functions encoding, say, numbers of subgroups of given index.

Over the following decades, a variety of methods have been developed and applied to predict the behaviour and to study symmetries of zeta functions associated with algebraic structures, and to produce explicit formulae. Theoretical work in this area often employs a blend of combinatorics and p-adic integration; see [42] for a survey. On the practical side, a range of effective methods have been devised, implemented (see [35]), and used to symbolically compute zeta functions of algebraic structures; see [31] and the references therein.

Variation of the prime: symbolic enumeration. A common feature of zeta functions $\zeta_G(s)$ attached to an algebraic structure G in the literature is that they often admit an Euler factorisation $\zeta_G(s) = \prod_p \zeta_{G,p}(s)$ into *local factors* indexed by primes p. For instance, when G is a nilpotent group and we are enumerating its subgroups, we can view $\zeta_{G,p}(s)$ as the ordinary generating function in p^{-s} encoding the numbers of subgroups of p-power index of G.

Results from p-adic integration often guarantee that local factors $\zeta_{G,p}(s)$ are rational in p^{-s}, i.e. of the form $\zeta_{G,p}(s) = W_p(p^{-s})$ for some $W_p(Y) \in \mathbb{Q}(Y)$; see, in particular, [22, Thm 1]. A key theme is then to study how the $W_p(Y)$ vary with the prime p. By the *symbolic enumeration* of the objects counted by $\zeta_G(s)$, we mean the task of providing a meaningful description of $W_p(Y)$ as a function of p. In a surprising number of cases of interest, deep *uniformity results* ensure the existence of a *single* bivariate rational function $W(X, Y)$ such that $\zeta_{G,p}(s) = W(p, p^{-s})$ for all primes p (perhaps ignoring a finite number of exceptions); see, for example, [2, Thm 1.1], [12, Thm 1.2], [36, Thm A], or [40, Thm B]. In such situations, understanding our zeta function is tantamount to understanding $W(X, Y)$.

In this context, permutation statistics (and, more generally, combinatorial objects such as graphs and posets) have recently found spectacular applications, in particular when it comes to describing the numerators of the rational functions $W(X, Y)$ from above; see, for instance, [2, 11–13, 15, 40]. Conversely, the need for combinatorial descriptions of such zeta functions gave rise to new directions in the study of permutation statistics and, more generally, combinatorial objects; see, e.g. [7, 8, 14, 16, 17, 19, 39].

Average sizes of kernels and orbits of groups. Introduced in [32] and developed further in [10, 33, 36], *ask zeta functions* are generating functions encoding average sizes of kernels in modules of matrices. One main motivation for studying these functions comes from group theory. Indeed, for groups with a sufficiently powerful Lie theory, the enumeration of linear orbits and conjugacy classes boils down to determining average sizes of kernels within matrix Lie algebras—this is essentially the orbit-counting lemma.

Amidst a plethora of algebraically-defined zeta functions, ask zeta functions stand out as particularly amenable to combinatorial methods. Indeed, natural operations at the level of the modules (or groups) often translate into natural operations of corresponding rational generating functions. In particular, ask zeta functions of (diagonal) direct sums of modules are Hadamard products of the ask zeta function of the summands; see [32, §3.4].

This work: overview. A first, somewhat coincidental, application of permutation statistics in the context of ask zeta functions appeared in [32, §5.4]; see also Remark 5.13. In the present article, we develop this connection between permutation statistics and ask zeta functions significantly further. In Sections 3–4, we introduce shuffle algebras attached to coloured descent statistics (based on [25]) and we relate them to Poirier's coloured quasisymmetric functions [29]. Along the way, in Section 4.2, we consider a triple of coloured permutation statistics which simultaneously keeps track of (coloured) descent numbers, (coloured) comajor indices, and colour multiplicities—this triple turns out to be shuffle compatible. Drawing upon the work of Gessel and Zhuang [20], in Theorem 4.2, we obtain an explicit embedding of the shuffle algebra associated with our triple into a power series ring with multiplication given by the Hadamard product.

Owing to the fact that the present work bridges two mathematical areas, in Section 2, we state Theorem 2.2, a self-contained and elementary form of Theorem 4.2 designed with a view towards applications to zeta functions. We will prove Theorem 2.2 in Section 4.3.

In Section 5, we then conclude our paper with applications to zeta functions. We use our results to produce explicit combinatorial formulae for Hadamard products within natural (infinite) families of zeta functions. In particular, Corollary 5.11 completely settles a question from [36] which asked for an interpretation of certain ask zeta functions attached to hypergraphs in terms of permutation statistics. As our main group-theoretic applications, we then obtain explicit formulae for zeta functions enumerating

(a) conjugacy classes in groups derived from direct products of free class-2-nilpotent groups (Corollary 5.14) and

(b) linear orbits of direct products of full unitriangular matrix groups (Corollary 5.15).

An extended abstract of the present article for the FPSAC 2024 conference will appear as [9]. Some results in Sections 3.3 and 4.2 follow (otherwise unpublished) results from the second author's PhD thesis [25].

Throughout this article, all rings and algebras will be assumed to be commutative and unital.

2 Hadamard products and coloured configurations

In this section, we provide relevant concepts and notation and we provide a self-contained account of our main result pertaining to Hadamard products of suitable rational generating functions. Its proof relies on the coloured shuffle compatibility of certain permutation statistics and the structure of associated coloured shuffle algebras. We will describe the latter in Section 4.

2.1 Coloured permutations

Coloured elements, sets, and integers. We consider the poset $\Gamma = \{0 > 1 > 2 > \cdots\}$, the elements of which we call **colours**. By a **coloured element** of a set T, we mean an expression t^c (formally: a pair (t, c)) for $t \in T$ and $c \in \Gamma$. By a **coloured subset** of T, we mean a set of the form $\{t^{\gamma(t)} : t \in S\}$ for a subset $S \subseteq T$ and a function $\gamma\colon S \to \Gamma$.

Let $\Sigma = \{1 < 2 < \cdots\}$ be the usual poset of positive integers. On the set of all coloured positive integers, we consider the total order

$$\cdots < 1^1 < 2^1 < \cdots < 1^0 < 2^0 < \cdots .$$

That is, $\sigma_1^{\gamma_1} < \sigma_2^{\gamma_2}$ if and only if $\gamma_1 = \gamma_2$ and $\sigma_1 < \sigma_2$, or if $\gamma_1 > \gamma_2$ in $\mathbb{Z}$ (equivalently: $\gamma_1 < \gamma_2$ in Γ). This is the **colour order** (see [3, §3]) and it corresponds to the left lexicographic order on $\Gamma \times \Sigma$.

Coloured permutations and descents. By a **coloured permutation** we mean a string $\boldsymbol{a} = \sigma^\gamma = \sigma_1^{\gamma_1} \cdots \sigma_n^{\gamma_n}$ for $n \geqslant 0$, distinct $\sigma_1, \ldots, \sigma_n \in \Sigma$, and arbitrary $\gamma_1, \ldots, \gamma_n \in \Gamma$. We write $|\boldsymbol{a}| = n$ for the **length** of $\boldsymbol{a}$. (There is a unique coloured permutation of length zero.) We further write $\operatorname{sym}(\boldsymbol{a}) = \{\sigma_1, \ldots, \sigma_n\}$ and $\operatorname{pal}(\boldsymbol{a}) = \{\gamma_1, \ldots, \gamma_n\}$ for the **set of symbols** and **palette** of $\boldsymbol{a}$, respectively. Finally, let $\operatorname{pal}^*(\boldsymbol{a}) = \operatorname{pal}(\boldsymbol{a}) \setminus \{0\}$. As usual, we write $[m] = \{1, \ldots, m\}$. The **descent set** of a coloured permutation $\boldsymbol{a} = \sigma^\gamma$ as above is

$$\operatorname{Des}(\boldsymbol{a}) = \begin{cases} \{i \in [n-1] : \sigma_i^{\gamma_i} > \sigma_{i+1}^{\gamma_{i+1}}\}, & \text{if } \gamma_1 = 0, \\ \{i \in [n-1] : \sigma_i^{\gamma_i} > \sigma_{i+1}^{\gamma_{i+1}}\} \cup \{0\}, & \text{otherwise.} \end{cases}$$

Here, the comparisons involve the colour order defined above. The **descent number** and **comajor index** are defined (as usual) as functions of the descent set: $\operatorname{des}(\boldsymbol{a}) = |\operatorname{Des}(\boldsymbol{a})|$ and $\operatorname{comaj}(\boldsymbol{a}) = \sum_{i \in \operatorname{Des}(\boldsymbol{a})} (n - i)$.

2.2 Labelled coloured configurations

Coloured configurations. Let $\mathcal{A}$ be the set of all coloured permutations, and let $\mathbb{N}_0\mathcal{A}$ be the free commutative monoid with basis $\mathcal{A}$. We call elements of $\mathbb{N}_0\mathcal{A}$ **coloured configurations**. These elements are of the form $f = \sum_{\boldsymbol{a} \in \mathcal{A}} f_{\boldsymbol{a}} \boldsymbol{a}$, where each $f_{\boldsymbol{a}}$ belongs to $\mathbb{N}_0$ and almost all $f_{\boldsymbol{a}}$ are zero. (Hence, coloured configurations and multisets of coloured permutations are identical concepts.) Write $\operatorname{supp}(f) = \{\boldsymbol{a} \in \mathcal{A} : f_{\boldsymbol{a}} \neq 0\}$, $\operatorname{sym}(f) = \bigcup_{\boldsymbol{a} \in \operatorname{supp}(f)} \operatorname{sym}(\boldsymbol{a})$, and $\operatorname{pal}^*(f) = \bigcup_{\boldsymbol{a} \in \operatorname{supp}(f)} \operatorname{pal}^*(\boldsymbol{a})$. We call $f, g \in \mathbb{N}_0\mathcal{A}$ **(symbol-)disjoint** if $\operatorname{sym}(f) \cap \operatorname{sym}(g) = \varnothing$; if, in addition, $\operatorname{pal}^*(f) \cap \operatorname{pal}^*(g) = \varnothing$, then f and g are **strongly disjoint**.

For disjoint $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{A}$, let $\boldsymbol{a} \sqcup\!\sqcup\, \boldsymbol{b} \in \mathbb{N}_0\mathcal{A}$ be the sum over all shuffles of $\boldsymbol{a}$ and $\boldsymbol{b}$. (Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are disjoint, these shuffles are themselves coloured permutations.) We bi-additively extend this to define $f \sqcup\!\sqcup\, g$ for disjoint $f, g \in \mathbb{N}_0\mathcal{A}$; observe that $\operatorname{sym}(f \sqcup\!\sqcup\, g) = \operatorname{sym}(f) \cup \operatorname{sym}(g)$ and $\operatorname{pal}(f \sqcup\!\sqcup\, g) = \operatorname{pal}(f) \cup \operatorname{pal}(g)$. (Hence, $\operatorname{pal}^*(f \sqcup\!\sqcup\, g) = \operatorname{pal}^*(f) \cup \operatorname{pal}^*(g)$.)

Labels. Let $\mathbb{U} = \{\pm X^k : k \in \mathbb{Z}\}$, viewed as a subgroup of the multiplicative group of the field $\mathbb{Q}(X)$. For $\alpha \colon \Gamma \to \mathbb{U}$, write $\operatorname{supp}(\alpha) = \{c \in \Gamma : \alpha(c) \neq 1\}$ and, for $\boldsymbol{a} = \sigma_1^{\gamma_1} \cdots \sigma_n^{\gamma_n}$ as above, let $\alpha(\boldsymbol{a}) = \prod_{i=1}^n \alpha(\gamma_i)$. A **labelled coloured configuration** is a pair (f, α), where $f \in \mathbb{N}_0\mathcal{A}$ and $\alpha \colon \Gamma \to \mathbb{U}$ satisfies $\operatorname{supp}(\alpha) \subseteq \operatorname{pal}^*(f)$.

Equivalence. Let (f, α) be a labelled coloured configuration. Let $\phi \colon \operatorname{sym}(f) \to S$ and $\psi \colon \operatorname{pal}^*(f) \to P$ be order-preserving bijections onto finite subsets of Σ and $\Gamma \setminus \{0\}$, respectively. Given ϕ and ψ, define a labelled coloured permutation (f', α') as follows. For $\boldsymbol{a} \in \operatorname{supp}(f)$, say $\boldsymbol{a} = \sigma_1^{\gamma_1} \cdots \sigma_n^{\gamma_n}$, write $\boldsymbol{a}' = \phi(\sigma_1)^{\psi(\gamma_1)} \cdots \phi(\sigma_n)^{\psi(\gamma_n)}$. Define $f' = \sum_{\boldsymbol{a} \in \operatorname{supp}(f)} f_{\boldsymbol{a}} \boldsymbol{a}'$. We define α' to be the function $\Gamma \to \mathbb{U}$ whose support is contained in P and which satisfies $\alpha'(\psi(c)) = \alpha(c)$ for $c \in \operatorname{pal}^*(f)$. We call (f, α) and each (f', α') (as ϕ and ψ range over possible choices) **equivalent**, written $(f, \alpha) \asymp (f', \alpha')$. This defines an equivalence relation on labelled coloured configurations.

Coherence. We say that labelled coloured configurations (f, α) and (g, β) are **coherent** if f and g are disjoint and if, in addition, $\alpha(c) = \beta(c)$ for all $c \in \mathrm{pal}^*(f) \cap \mathrm{pal}^*(g)$. In that case, we may define $\alpha \cup \beta \colon \Gamma \to \mathbb{U}$ via

$$(\alpha \cup \beta)(c) = \begin{cases} \alpha(c), & \text{if } c \in \mathrm{pal}^*(f), \\ \beta(c), & \text{if } c \in \mathrm{pal}^*(g), \\ 1, & \text{otherwise.} \end{cases}$$

The pair $(f \sqcup g, \alpha \cup \beta)$ is then a labelled coloured configuration too. (Indeed, we have $\mathrm{supp}(\alpha \cup \beta) = \mathrm{supp}(\alpha) \cup \mathrm{supp}(\beta) \subseteq \mathrm{pal}^*(f) \cup \mathrm{pal}^*(g) = \mathrm{pal}^*(f \sqcup g)$.)

Note that if (f, α) and (g, β) are labelled coloured configurations such that f and g are strongly disjoint, then (f, α) and (g, β) are automatically coherent and, moreover, $\alpha \cup \beta = \alpha\beta$ is the pointwise product of α and β.

2.3 Rational functions and statement of the main result

Given a labelled coloured configuration (f, α) and $\varepsilon \in \mathbb{Z}$, we define a rational formal power series

$$W^{\varepsilon}_{f,\alpha} = W^{\varepsilon}_{f,\alpha}(X, Y) = \sum_{\boldsymbol{a} \in \mathrm{supp}(f)} f_{\boldsymbol{a}} \frac{\alpha(\boldsymbol{a}) X^{\varepsilon\, \mathrm{comaj}(\boldsymbol{a})} Y^{\mathrm{des}(\boldsymbol{a})}}{(1 - Y)(1 - X^{\varepsilon} Y) \cdots (1 - X^{\varepsilon |\boldsymbol{a}|} Y)} \in \mathbb{Q}(X)[\![Y]\!].$$

Note that, by construction, if $(f, \alpha) \asymp (f', \alpha')$, then $W^{\varepsilon}_{f,\alpha} = W^{\varepsilon}_{f',\alpha'}$ for all $\varepsilon \in \mathbb{Z}$.

Example 2.1. Let $f = 1^0 + 1^1$. Let $\alpha \colon \Gamma \to \mathbb{U}$ with $\mathrm{supp}(\alpha) \subseteq \mathrm{pal}^*(f) = \{1\}$. Then

$$W^{\varepsilon}_{f,\alpha} = \frac{1 + \alpha(1) X^{\varepsilon} Y}{(1 - Y)(1 - X^{\varepsilon} Y)}.$$

Recall that the **Hadamard product** of two formal power series $A(Y) = \sum_{k=0}^{\infty} a_k Y^k$ and $B(Y) = \sum_{k=0}^{\infty} b_k Y^k$ (with coefficients in a common ring) is the power series $A(Y) *_Y B(Y) = \sum_{k=0}^{\infty} a_k b_k Y^k$. The following is the key to our explicit computations of Hadamard products in this paper.

Theorem 2.2. *Let (f, α) and (g, β) be coherent labelled coloured configurations. Then*

$$W^{\varepsilon}_{f,\alpha} *_Y W^{\varepsilon}_{g,\beta} = W^{\varepsilon}_{f \sqcup g,\, \alpha \cup \beta}$$

for each $\varepsilon \in \mathbb{Z}$.

We will prove Theorem 2.2 in Section 4.3.

Example 2.3. Let $f = 1^0 + 1^1$ and $g = 2^0 + 2^2$. Then

$$\begin{aligned} f \sqcup g &= (1^0 + 1^1) \sqcup (2^0 + 2^2) \\ &= 1^0 \sqcup 2^0 + 1^0 \sqcup 2^2 + 1^1 \sqcup 2^0 + 1^1 \sqcup 2^2 \\ &= 1^0 2^0 + 2^0 1^0 + 1^0 2^2 + 2^2 1^0 + 1^1 2^0 + 2^0 1^1 + 1^1 2^2 + 2^2 1^1. \end{aligned}$$

The descent numbers of comajor indices of the coloured permutations appearing as summands above are recorded in the following table:

	$1^0 2^0$	$2^0 1^0$	$1^0 2^2$	$2^2 1^0$	$1^1 2^0$	$2^0 1^1$	$1^1 2^2$	$2^2 1^1$
des	0	1	1	1	1	1	2	1
comaj	0	1	1	2	2	1	3	2

Let α and β satisfy $\mathrm{supp}(\alpha) \subseteq \{1\}$ and $\mathrm{supp}(\beta) \subseteq \{2\}$. Then, by Theorem 2.2,

$$\begin{aligned} W^{\varepsilon}_{f,\alpha} *_Y W^{\varepsilon}_{g,\beta} &= \frac{1+\alpha(1)X^{\varepsilon}Y}{(1-Y)(1-X^{\varepsilon}Y)} *_Y \frac{1+\beta(2)X^{\varepsilon}Y}{(1-Y)(1-X^{\varepsilon}Y)} \\ &= \frac{1+(1+\alpha(1)+\beta(2))X^{\varepsilon}Y+(\alpha(1)+\beta(2)+\alpha(1)\beta(2))X^{2\varepsilon}Y+\alpha(1)\beta(2)X^{3\varepsilon}Y^2}{(1-Y)(1-X^{\varepsilon}Y)(1-X^{2\varepsilon}Y)} \\ &= W^{\varepsilon}_{f\sqcup g,\,\alpha\cup\beta}. \end{aligned}$$

Corollary 2.4. *Let $\varepsilon \in \mathbb{Z}$ be fixed. Then the set*

$$\left\{ W^{\varepsilon}_{f,\alpha} : (f,\alpha) \text{ is a labelled coloured configuration} \right\}$$

is closed under Hadamard products in Y.

Proof. Given coloured configurations (f,α) and (g,β), we can find (g',β') such that f and g' are strongly disjoint and $(g,\beta) \asymp (g',\beta')$. In that case, $W^{\varepsilon}_{f,\alpha} *_Y W^{\varepsilon}_{g,\beta} = W^{\varepsilon}_{f,\alpha} *_Y W^{\varepsilon}_{g',\beta'} = W^{\varepsilon}_{f\sqcup g',\,\alpha\beta'}$ is computed by Theorem 2.2. ∎

In Section 5, we will apply Theorem 2.2 to provide explicit formulae for Hadamard products of ask, class- and orbit-counting zeta functions.

3 Coloured quasisymmetric functions

For technical reasons, in this and the following section, we will only consider coloured permutations with colours drawn from $\{0 > 1 > \cdots > r-1\} \subset \Gamma$. For clarity, we occasionally refer to these as r**-coloured permutations**. (We similarly refer to r-coloured integers, etc.)

3.1 Coloured quasisymmetric functions and descent sets

Coloured quasisymmetric functions. Let $x_i^{(j)}$ for $i = 1, 2, \ldots$ and $j = 0, 1, \ldots, r-1$ be independent (commuting) variables. We write $\boldsymbol{x}^{(j)} = (x_1^{(j)}, x_2^{(j)}, \ldots)$. The **coloured quasisymmetric function** attached to an r-coloured permutation $\boldsymbol{a} = \sigma^{\gamma}$ of length n is

$$F_{\boldsymbol{a}} = F_{\boldsymbol{a}}(\boldsymbol{x}^{(0)}, \ldots, \boldsymbol{x}^{(r-1)}) = \sum_{\substack{1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n \\ j \in \mathrm{Des}^*(\boldsymbol{a}) \Rightarrow i_j < i_{j+1}}} x_{i_1}^{(\gamma_1)} x_{i_2}^{(\gamma_2)} \cdots x_{i_n}^{(\gamma_n)}, \tag{3.1}$$

where $\mathrm{Des}^*(\boldsymbol{a}) = \mathrm{Des}(\boldsymbol{a}) \setminus \{0\}$. This is a (homogeneous) formal power series of degree n in the variables $\boldsymbol{x}^{(0)}, \ldots, \boldsymbol{x}^{(r-1)}$. These functions were first introduced in [29]; see also [23, 25, 27]. The space $\mathrm{QSym}^{(r)}$ spanned by all such coloured quasisymmetric functions naturally forms a $\mathbb{Q}$-algebra.

The coloured descent set I. It is clear that the coloured quasisymmetric function $F_{\boldsymbol{a}}$ attached to $\boldsymbol{a} = \sigma^\gamma$ (of length n) only depends on the descent set $\mathrm{Des}(\boldsymbol{a})$ and on γ, rather than on the r-coloured permutation itself. Both $\mathrm{Des}(\boldsymbol{a})$ and γ can be extracted from the **coloured descent set** (cf. [1, Definition 2.2]) of $\boldsymbol{a}$ which is defined, for $n > 0$, by

$$\mathrm{sDes}(\boldsymbol{a}) = \{i^{\gamma_i} : i \in [n-1],\, \gamma_i \neq \gamma_{i+1} \text{ or } (\gamma_i = \gamma_{i+1} \text{ and } \sigma_i > \sigma_{i+1})\} \cup \{n^{\gamma_n}\};$$

for $n = 0$, we let $\mathrm{sDes}(\boldsymbol{a}) = \varnothing$. In any case, $\mathrm{sDes}(\boldsymbol{a})$ is an r-coloured subset of $[n]$.

We write $\mathbb{S}^{(r)}$ for the set of all r-coloured subsets of the set of positive integers. We define $\sup_*(\varnothing) = 0$; for a non-empty $A \in \mathbb{S}^{(r)}$, say $A = \{a_1^{\gamma_1}, \dots, a_k^{\gamma_k}\}$ with $a_1 < \cdots < a_k$, we define $\sup_*(A) = a_k$. Each element of $\mathbb{S}^{(r)}$ is of the form $\mathrm{sDes}(\boldsymbol{a})$ for some r-coloured permutation $\boldsymbol{a}$; note that $|\boldsymbol{a}| = \sup_*(\mathrm{sDes}(\boldsymbol{a}))$.

Since $F_{\boldsymbol{a}} = F_{\boldsymbol{b}}$ whenever $\mathrm{sDes}(\boldsymbol{a}) = \mathrm{sDes}(\boldsymbol{b})$, we also write F_A for the quasisymmetric function attached to any r-coloured permutation $\boldsymbol{a}$ with $\mathrm{sDes}(\boldsymbol{a}) = A$.

3.2 Coloured permutation statistics

A **coloured permutation statistic** is a function st defined on the set of coloured permutations such that given a coloured permutation σ^γ, if π is a permutation of the same length as σ and with the same relative order, then $\mathrm{st}(\sigma^\gamma) = \mathrm{st}(\pi^\gamma)$. Given coloured permutation statistics $\mathrm{st}_1, \dots, \mathrm{st}_k$, we regard the tuple $(\mathrm{st}_1, \dots, \mathrm{st}_k)$ as a coloured permutation statistic via $(\mathrm{st}_1, \dots, \mathrm{st}_k)(\boldsymbol{a}) = (\mathrm{st}_1(\boldsymbol{a}), \dots, \mathrm{st}_k(\boldsymbol{a}))$. Given a coloured permutation $\boldsymbol{a} = \sigma^\gamma = \sigma_1^{\gamma_1} \cdots \sigma_n^{\gamma_n}$, let $\mathrm{col}_j(\boldsymbol{a}) := |\{i \in [n] : \gamma_i = j\}|$. The **colour vector** of a $\boldsymbol{a}$ is $\mathbf{col}(\boldsymbol{a}) = (\mathrm{col}_0(\boldsymbol{a}), \dots, \mathrm{col}_{r-1}(\boldsymbol{a}))$; this is a weak composition of n. The functions $\mathbf{col}$, des, and comaj are coloured permutation statistics.

3.3 Coloured shuffle compatibility and shuffle algebras

Shuffle compatibility. Let st be a coloured permutation statistic. Following [20, 25], we say that st is **shuffle compatible** if for all disjoint coloured permutations $\boldsymbol{a}$ and $\boldsymbol{b}$, the multiset $\{\{\mathrm{st}(\boldsymbol{c}) : \boldsymbol{c} \in \boldsymbol{a} \sqcup\!\sqcup \boldsymbol{b}\}\}$ only depends on $\mathrm{st}(\boldsymbol{a}), \mathrm{st}(\boldsymbol{b})$ and the lengths of $\boldsymbol{a}$ and $\boldsymbol{b}$. Here, as before, $\boldsymbol{a} \sqcup\!\sqcup \boldsymbol{b}$ denotes the set of all coloured permutations obtained as shuffles of $\boldsymbol{a}$ and $\boldsymbol{b}$.

Shuffle algebras. Generalising [20, 25], we associate a **shuffle algebra** $\mathcal{A}^{(r)}_{\mathrm{st}}$ over $\mathbb{Q}$ to a shuffle-compatible coloured permutation statistic st as follows. First, st defines an equivalence relation $\sim_{\mathrm{st}}$ on r-coloured permutations via $\boldsymbol{a} \sim_{\mathrm{st}} \boldsymbol{b}$ if and only if $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same length and $\mathrm{st}(\boldsymbol{a}) = \mathrm{st}(\boldsymbol{b})$; we refer to this as st-**equivalence**. We write $[\boldsymbol{a}]_{\mathrm{st}}$ to denote the st-equivalence class of $\boldsymbol{a}$. As a $\mathbb{Q}$-vector space $\mathcal{A}^{(r)}_{\mathrm{st}}$ has a basis given by the st-equivalence classes of r-coloured permutations. The multiplication is given by linearly extending the rule

$$[\boldsymbol{a}]_{\mathrm{st}}\, [\boldsymbol{b}]_{\mathrm{st}} = \sum_{\boldsymbol{c} \in \boldsymbol{a} \sqcup\!\sqcup \boldsymbol{b}} [\boldsymbol{c}]_{\mathrm{st}},$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are r-coloured permutations on disjoint sets of symbols. (Thanks to the shuffle compatibility of st, this yields a well-defined multiplication on $\mathcal{A}^{(r)}_{\mathrm{st}}$.)

Let st and st′ be shuffle compatible. Suppose that st **refines** st′ in the sense that $\mathrm{st}'(\boldsymbol{a}) = \mathrm{st}'(\boldsymbol{b})$ whenever $\mathrm{st}(\boldsymbol{a}) = \mathrm{st}(\boldsymbol{b})$. Then the rule $[\boldsymbol{a}]_{\mathrm{st}} \mapsto [\boldsymbol{a}]_{\mathrm{st}'}$ defines a surjective $\mathbb{Q}$-algebra homomorphism $\mathcal{A}^{(r)}_{\mathrm{st}} \twoheadrightarrow \mathcal{A}^{(r)}_{\mathrm{st}'}$.

The coloured descent set II. The following is a restatement of [23, Eqn (3.4)] (see also [25, Thm 4.2.4]. It implies that the coloured descent set sDes is shuffle compatible and allows us to identify $\mathrm{QSym}^{(r)}$ and $\mathcal{A}^{(r)}_{\mathrm{sDes}}$.

Theorem 3.1. *Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be symbol-disjoint coloured permutations. Let* $\mathrm{sDes}(\boldsymbol{a}) = A$*,* $\mathrm{sDes}(\boldsymbol{b}) = B$*, and $n = |\boldsymbol{a}| + |\boldsymbol{b}|$. For an r-coloured subset C of $[n]$, let $\nu^C_{A,B}$ be the number of r-coloured permutations $\boldsymbol{c} \in \boldsymbol{a} \sqcup \boldsymbol{b}$ such that* $\mathrm{sDes}(\boldsymbol{c}) = C$*. Then*

$$F_A F_B = \sum_C \nu^C_{A,B} F_C,$$

where C runs over all r-coloured subsets of $[n]$.

Corollary 3.2 (Cf. [25, Thm 4.4.1])**.**

(i) The coloured descent set sDes *is shuffle compatible.*

(ii) The linear map on $\mathcal{A}^{(r)}_{\mathrm{sDes}}$ defined by

$$[\boldsymbol{a}]_{\mathrm{sDes}} \mapsto F_{\boldsymbol{a}}$$

is a $\mathbb{Q}$-algebra isomorphism $\mathcal{A}^{(r)}_{\mathrm{sDes}} \to \mathrm{QSym}^{(r)}$. ∎

4 Descent statistics and coloured shuffle compatibility

As in Section 3, we assume that all coloured permutations, sets, etc. are r-coloured for some arbitrary but fixed $r \gg 0$.

4.1 Coloured descent statistics

As a coloured version of [20, §2.1], we say that a coloured permutation statistic st is a **coloured descent statistic** if for all coloured permutations $\boldsymbol{a}$ and $\boldsymbol{b}$, we have $\mathrm{st}(\boldsymbol{a}) = \mathrm{st}(\boldsymbol{b})$ whenever $\mathrm{sDes}(\boldsymbol{a}) = \mathrm{sDes}(\boldsymbol{b})$. For $A \in \mathbb{S}^{(r)}$, we may then unambiguously define $\mathrm{st}(A) := \mathrm{st}(\boldsymbol{a})$ where $\boldsymbol{a}$ is any r-coloured permutation with $\mathrm{sDes}(\boldsymbol{a}) = A$. We write $\mathrm{st}(\mathbb{S}^{(r)}) = \{\mathrm{st}(A) : A \in \mathbb{S}^{(r)}\}$.

The following result and its proof constitute a coloured variant of [20, Thm 4.3] (cf. also [25, Lemma 4.4.4]). In Theorem 4.2, we will use it to prove that $(\mathrm{des}, \mathrm{comaj}, \mathbf{col})$ is shuffle compatible.

Lemma 4.1. *Let* st *be a coloured descent statistic. Then* st *is shuffle compatible if and only if there exist a $\mathbb{Q}$-algebra $\mathcal{B}$ and a $\mathbb{Q}$-algebra homomorphism $\phi\colon \mathrm{QSym}^{(r)} \to \mathcal{B}$ with the following properties.*

(i) For all $A, B \in \mathbb{S}^{(r)}$ with $\mathrm{st}(A) = \mathrm{st}(B)$ and $\sup_*(A) = \sup_*(B)$ (see Section 3.1), we have $\phi(F_A) = \phi(F_B)$.

(ii) For each (finite or infinite) sequence $A_1, A_2, \ldots \in \mathbb{S}^{(r)}$ such that the $(\mathrm{st}(A_i), \sup_*(A_i))$ are pairwise distinct, the images $\phi(F_{A_1}), \phi(F_{A_2}), \ldots$ are $\mathbb{Q}$-linearly independent.

In this case, the rule $[\boldsymbol{a}]_{\mathrm{st}} \mapsto \phi(F_{\boldsymbol{a}})$ yields an injective $\mathbb{Q}$-algebra homomorphism $\mathcal{A}^{(r)}_{\mathrm{st}} \to \mathcal{B}$.

Proof. If st is shuffle compatible, then Corollary 3.2 yields a homomorphism

$$\phi_{\mathrm{st}}\colon \operatorname{QSym}^{(r)} \xrightarrow{\cong} \mathcal{A}^{(r)}_{\mathrm{sDes}} \twoheadrightarrow \mathcal{A}^{(r)}_{\mathrm{st}} \tag{4.1}$$

such that $\phi_{\mathrm{st}}(F_{\boldsymbol{a}}) = [\boldsymbol{a}]_{\mathrm{st}}$ for each coloured permutation $\boldsymbol{a}$. By construction, ϕ_{st} has the desired properties. Conversely, suppose that ϕ satisfies (i)–(ii). Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be (symbol-)disjoint coloured permutations. Using Corollary 3.2, we obtain

$$\phi(F_{\boldsymbol{a}})\phi(F_{\boldsymbol{b}}) = \phi(F_{\boldsymbol{a}}F_{\boldsymbol{b}}) = \phi\left(\sum_{\boldsymbol{c}\in\boldsymbol{a}\sqcup\boldsymbol{b}} F_{\boldsymbol{c}}\right) = \sum_{\boldsymbol{c}\in\boldsymbol{a}\sqcup\boldsymbol{b}} \phi(F_{\boldsymbol{c}}). \tag{4.2}$$

Conditions (i)–(ii) allow us to recover the multiset $\{\{\mathrm{st}(\boldsymbol{c}) : \boldsymbol{c} \in \boldsymbol{a} \sqcup \boldsymbol{b}\}\}$ from the quadruple $(\mathrm{st}(\boldsymbol{a}), \mathrm{st}(\boldsymbol{b}), |\boldsymbol{a}|, |\boldsymbol{b}|)$. Indeed, the multiplicity of $\mathrm{st}(\boldsymbol{c})$ in our multiset is the coefficient of $\phi(F_{\boldsymbol{c}})$ in (4.2) We conclude that st is shuffle compatible.

Finally, let st be shuffle compatible and let ϕ with (i)–(ii) be given. Define ϕ_{st} as in (4.1). Then $\phi_{\mathrm{st}}(\boldsymbol{a}) = \phi_{\mathrm{st}}(\boldsymbol{b})$ if and only if $|\boldsymbol{a}| = |\boldsymbol{b}|$ and $\mathrm{st}(\boldsymbol{a}) = \mathrm{st}(\boldsymbol{b})$ if and only if $\phi(F_{\boldsymbol{a}}) = \phi(F_{\boldsymbol{b}})$. The kernel of ϕ_{st} is generated by all $F_{\boldsymbol{a}} - F_{\boldsymbol{b}}$ with $|\boldsymbol{a}| = |\boldsymbol{b}|$ and $\mathrm{st}(\boldsymbol{a}) = \mathrm{st}(\boldsymbol{b})$. In particular, $\mathrm{Ker}(\phi_{\mathrm{st}}) \subseteq \mathrm{Ker}(\phi)$ and the rule $[\boldsymbol{a}]_{\mathrm{st}} \mapsto F_{\boldsymbol{a}}$ indeed defines a $\mathbb{Q}$-algebra homomorphism $\psi\colon \mathcal{A}^{(r)}_{\mathrm{st}} \to \mathcal{B}$ with $\phi = \psi \circ \phi_{\mathrm{st}}$. By (ii), ψ is injective. ■

4.2 The shuffle algebra of $(\mathrm{des}, \mathrm{comaj}, \mathbf{col})$

The main shuffle algebra of interest to us is the one attached to $(\mathrm{des}, \mathrm{comaj}, \mathbf{col})$. The latter statistic is naturally refined by the coloured descent set sDes. Let $p_0, \dots, p_{r-1}, x, t$ be commuting variables over $\mathbb{Q}$. Write $\boldsymbol{p} = (p_0, \dots, p_{r-1})$ and $\boldsymbol{p}^{\boldsymbol{v}} = p_0^{v_0} \cdots p_{r-1}^{v_{r-1}}$. For a ring R, let $R[\![t*]\!]$ denote the ring $R[\![t]\!]$ with multiplication given by the Hadamard product in t.

Theorem 4.2.

(i) The tuple of statistics $(\mathrm{des}, \mathrm{comaj}, \mathbf{col})$ *is shuffle compatible.*

(ii) The linear map $H\colon \mathcal{A}^{(r)}_{(\mathrm{des},\mathrm{comaj},\mathbf{col})} \to \mathbb{Q}[\boldsymbol{p}, x][\![t*]\!]$ *defined by*

$$[\boldsymbol{a}]_{(\mathrm{des},\mathrm{comaj},\mathbf{col})} \mapsto \frac{\boldsymbol{p}^{\mathbf{col}(\boldsymbol{a})} x^{\mathrm{comaj}(\boldsymbol{a})} t^{\mathrm{des}(\boldsymbol{a})}}{(1-t)(1-xt)\cdots(1-x^{|\boldsymbol{a}|}t)} \tag{4.3}$$

is an injective algebra homomorphism.

Our proof of Theorem 4.2 will be based on the following strategy. First, we construct a sequence of judicious specialisation homomorphisms ψ_m defined on $\operatorname{QSym}^{(r)}$. Next, these ψ_m can be combined to form a single homomorphism $\operatorname{QSym}^{(r)} \to \mathbb{Q}[\boldsymbol{p}, x][\![t*]\!]$ which sends each $F_{\boldsymbol{a}}$ to the right-hand side of (4.3). Both parts of Theorem 4.2 then follow by invoking Lemma 4.1. This strategy and its execution are inspired by [20, Thm 4.5] and [25, Thm 4.4.3]. In particular, our Proposition 4.3 is analogous to [25, Eqn (2.12)]. (For related but coarser results, see [26,27].) Our Theorem 4.2 can be seen as a refined version of [25, Thm 4.4.3]. We note that using a suitable coloured version of [20, Lemma 3.6], we could exchange the comajor indices in the present article for the major indices used in [25], thus obtaining results that are more direct refinements of those from [25].

Let $\psi_m \colon \mathrm{QSym}^{(r)} \to \mathbb{Q}[\boldsymbol{p}, x]$ be the specialisation defined by the substitution

$$\begin{cases} x_i^{(0)} \leftarrow x^{i-1} p_0, & \text{if } 1 \leqslant i \leqslant m, \\ x_i^{(j)} \leftarrow x^{i-1} p_j, & \text{if } 1 < i \leqslant m \text{ and } 1 \leqslant j \leqslant r-1, \\ x_i^{(j)} \leftarrow 0, & \text{otherwise.} \end{cases}$$

This is well defined since all but finitely many variables are sent to zero. Note that each ψ_m is an algebra homomorphism. Define a $\mathbb{Q}$-linear map $\Psi \colon \mathrm{QSym}^{(r)} \to \mathbb{Q}[\boldsymbol{p}, x][\![t*]\!]$ via

$$\Psi(F_{\boldsymbol{a}}) = \sum_{m=1}^{\infty} \psi_m(F_{\boldsymbol{a}}) t^{m-1}.$$

Clearly, Ψ is an algebra homomorphism.

Proposition 4.3. *For each r-coloured permutation $\boldsymbol{a}$, we have*

$$\Psi(F_{\boldsymbol{a}}) = \frac{\boldsymbol{p}^{\mathbf{col}(\boldsymbol{a})} x^{\mathrm{comaj}(\boldsymbol{a})} t^{\mathrm{des}(\boldsymbol{a})}}{(1-t)(1-xt)\cdots(1-x^n t)}. \tag{4.4}$$

Proof. Let $\boldsymbol{a} = \sigma^\gamma$ be an r-coloured permutation of length n. When applying ψ_m to $F_{\boldsymbol{a}}$, a summand $s := x_{i_1}^{(\gamma_1)} x_{i_2}^{(\gamma_2)} \cdots x_{i_n}^{(\gamma_n)}$ is sent to zero if and only if $i_j > m$ for some j or both $i_1 = 1$ and $\gamma_1 \neq 0$. Writing $i_0 = 1$ from now on, we can equivalently express the latter condition as $0 \in \mathrm{Des}(\boldsymbol{a})$ and $i_0 \not< i_1$. In all other cases, s is sent to $\boldsymbol{p}^{\mathbf{col}(\boldsymbol{a})} x^{i_1 + \cdots + i_n - n}$. Using our convention $i_0 = 1$, we conclude that

$$\psi_m(F_{\boldsymbol{a}}) = \sum_{\substack{1 = i_0 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n \leqslant m \\ j \in \mathrm{Des}(\boldsymbol{a}) \Rightarrow i_j < i_{j+1}}} \boldsymbol{p}^{\mathbf{col}(\boldsymbol{a})} x^{i_1 + i_2 + \cdots + i_n - n}. \tag{4.5}$$

In order to eliminate the strict inequalities on the right-hand side of (4.5), we write

$$\begin{aligned} i'_n &= i_n - \chi_0 - \chi_1 - \cdots - \chi_{n-1}, \\ i'_{n-1} &= i_{n-1} - \chi_0 - \chi_1 - \cdots - \chi_{n-2}, \\ &\vdots \\ i'_2 &= i_2 - \chi_0 - \chi_1, \\ i'_1 &= i_1 - \chi_0, \end{aligned}$$

where

$$\chi_i := \begin{cases} 1, & \text{if } i \in \mathrm{Des}(\boldsymbol{a}), \\ 0, & \text{otherwise.} \end{cases}$$

In (4.5), since $i_1 > 1$ whenever $0 \in \mathrm{Des}(\boldsymbol{a})$, we have $i'_1 \geqslant 1$. Next,

$$i_1 + i_2 + \cdots + i_n = i'_1 + i'_2 + \cdots + i'_n + \sum_{i=0}^{n-1} (n-i)\chi_i$$

and

$$\sum_{i=0}^{n-1} (n-i)\chi_i = \sum_{i \in \mathrm{Des}(\boldsymbol{a})} (n-i) = \mathrm{comaj}(\boldsymbol{a}).$$

Equation (4.5) therefore becomes

$$\psi_m(F_{\boldsymbol{a}}) = \sum_{1\leqslant i_1'\leqslant i_2'\leqslant\cdots\leqslant i_n'\leqslant m-\operatorname{des}(\boldsymbol{a})} \boldsymbol{p}^{\mathbf{col}(\boldsymbol{a})} x^{i_1'+i_2'+\cdots+i_n'-n+\operatorname{comaj}(\boldsymbol{a})} \tag{4.6}$$

and the claim follows easily from the identity of formal power series

$$\sum_{1\leqslant j_1\leqslant\cdots\leqslant j_n} \lambda_1^{j_1}\cdots\lambda_n^{j_n} = \frac{\lambda_1\cdots\lambda_n}{(1-\lambda_1\cdots\lambda_n)\cdots(1-\lambda_{n-1}\lambda_n)(1-\lambda_n)}. \qquad \blacksquare$$

Lemma 4.4. *Let $\boldsymbol{a}_1,\dots,\boldsymbol{a}_k$ be r-coloured permutations such that the triples*

$$(\operatorname{des}(\boldsymbol{a}_i), \operatorname{comaj}(\boldsymbol{a}_i), \mathbf{col}(\boldsymbol{a}_i))$$

are pairwise distinct. Then $\Psi(\boldsymbol{a}_1),\dots,\Psi(\boldsymbol{a}_k)$ are linearly independent over $\mathbb{Q}$.

Proof. First note that for an r-coloured permutation $\boldsymbol{b}$, we have

$$\Psi(F_{\boldsymbol{b}}) = \boldsymbol{p}^{\mathbf{col}(\boldsymbol{b})} x^{\operatorname{comaj}(\boldsymbol{b})} t^{\operatorname{des}(\boldsymbol{b})} + \mathcal{O}(t^{\operatorname{des}(\boldsymbol{b})+1}). \tag{4.7}$$

Suppose, for the sake of contradiction, that the claim is false. Choose k minimal such that $\Psi(\boldsymbol{a}_1),\dots,\Psi(\boldsymbol{a}_k)$ are linearly dependent, say $\sum_{i=1}^k \lambda_i\Psi(\boldsymbol{a}_i) = 0$ for nonzero $\lambda_i \in \mathbb{Q}$. Let $d = \min(\operatorname{des}(\boldsymbol{a}_1),\dots,\operatorname{des}(\boldsymbol{a}_k))$. After suitably permuting indices, we may assume that for some $1 \leqslant \ell \leqslant k$, we have $\operatorname{des}(\boldsymbol{a}_i) = d$ if and only if $i \leqslant \ell$. Then (4.7) shows that

$$0 \equiv \sum_{i=1}^k \lambda_i\Psi(\boldsymbol{a}_i) \equiv \sum_{i=1}^\ell \lambda_i\Psi(\boldsymbol{a}_i) \equiv \sum_{i=1}^\ell \lambda_i \boldsymbol{p}^{\mathbf{col}(\boldsymbol{a}_i)} x^{\operatorname{comaj}(\boldsymbol{a}_i)} t^d \pmod{t^{d+1}}$$

and thus $\sum_{i=1}^\ell \lambda_i \boldsymbol{p}^{\mathbf{col}(\boldsymbol{a}_i)} x^{\operatorname{comaj}(\boldsymbol{a}_i)} = 0$. By assumption, the $(\mathbf{col}(\boldsymbol{a}_i), \operatorname{comaj}(\boldsymbol{a}_i))$ are pairwise distinct for $i = 1,\dots,\ell$. As distinct monomials are linearly independent, we conclude that $\lambda_1 = \cdots = \lambda_\ell = 0$, contradicting the minimality of our counterexample. $\blacksquare$

Proof of Theorem 4.2. Equation (4.4) and Lemma 4.4 show that Ψ satisfies the conditions on ϕ in Lemma 4.1(i) and (ii), respectively. This concludes the proof. $\blacksquare$

We record the following alternative form of Theorem 4.2(ii) which more closely resembles its ancestor [20, Thm 4.15(c)] and relative [25, Thm 4.4.3]. The map $\tilde{H}$ in Corollary 4.5 only differs from H in Theorem 4.2 in the factor $z^{|\boldsymbol{a}|}$ and an additional factor t for $|\boldsymbol{a}| > 0$.

Corollary 4.5. *The linear map $\tilde{H}\colon \mathcal{A}^{(r)}_{(\operatorname{des},\operatorname{comaj},\mathbf{col})} \to \mathbb{Q}[\boldsymbol{p},x,z][\![t*]\!]$ defined by*

$$[\boldsymbol{a}]_{(\operatorname{des},\operatorname{comaj},\mathbf{col})} \mapsto \begin{cases} \frac{\boldsymbol{p}^{\mathbf{col}(\boldsymbol{a})} x^{\operatorname{comaj}(\boldsymbol{a})} t^{\operatorname{des}(\boldsymbol{a})+1}}{(1-t)(1-xt)\cdots(1-x^{|\boldsymbol{a}|}t)} z^{|\boldsymbol{a}|}, & \text{if } |\boldsymbol{a}| \geqslant 1 \\ \frac{1}{1-t}, & \text{if } |\boldsymbol{a}| = 0 \end{cases}$$

is an injective algebra homomorphism.

Proof. We write $[\cdot]$ instead of $[\cdot]_{(\operatorname{des},\operatorname{comaj},\mathbf{col})}$. Consider the injective homomorphism $\mu\colon \mathbb{Q}[\boldsymbol{p},x] \to \mathbb{Q}[\boldsymbol{p},x,z]$ which fixes x and sends each p_i to $p_i z$. Let M be the injective homomorphism

$\mathbb{Q}[\boldsymbol{p},x][\![t\ast]\!] \to \mathbb{Q}[\boldsymbol{p},x,z][\![t\ast]\!]$ with $M(\sum_{m=0}^{\infty} a_m t^m) = \sum_{m=0}^{\infty} \mu(a_m)t^m$. Let H be as in Theorem 4.2. Then $J := M \circ H$ is the injective homomorphism which sends each $[\boldsymbol{a}]$ to

$$\frac{\boldsymbol{p}^{\mathbf{col}(\boldsymbol{a})} x^{\mathrm{comaj}(\boldsymbol{a})} t^{\mathrm{des}(\boldsymbol{a})}}{(1-t)(1-xt)\cdots(1-x^{|\boldsymbol{a}|}t)} z^{|\boldsymbol{a}|}.$$

By construction, $\tilde{H}([\boldsymbol{a}]) = t \cdot J([\boldsymbol{a}])$ for $|\boldsymbol{a}| > 0$. Then $\tilde{H}$ and $t \cdot J$ agree on the $\mathbb{Q}$-span of all $[\boldsymbol{a}]$ with $|\boldsymbol{a}| > 0$. We thus conclude that for $|\boldsymbol{a}|, |\boldsymbol{b}| > 0$, we have

$$\tilde{H}([\boldsymbol{a}][\boldsymbol{b}]) = t \cdot J([\boldsymbol{a}][\boldsymbol{b}]) = t \cdot (J([\boldsymbol{a}]) \ast_t J([\boldsymbol{b}])) = (t \cdot J([\boldsymbol{a}])) \ast_t (t \cdot J([\boldsymbol{b}])) = \tilde{H}([\boldsymbol{a}]) \ast_t \tilde{H}([\boldsymbol{b}]).$$

■

4.3 Proof of Theorem 2.2

We again write $[\cdot]$ instead of $[\cdot]_{(\mathrm{des},\mathrm{comaj},\mathbf{col})}$. Let (f,α) and (g,β) be coherent labelled coloured configurations as in Section 2. Choose $r \gg 0$ such that the coloured permutations appearing in f and g only involve colours from $\{0,\dots,r-1\}$. Let H be as in Theorem 4.2. Define $\rho_0,\dots,\rho_{r-1} \in \mathbb{Q}(X)$ via

$$\rho_i = \begin{cases} \alpha(i), & \text{if } i \in \mathrm{pal}^*(f), \\ \beta(i), & \text{if } i \in \mathrm{pal}^*(g), \\ 1, & \text{otherwise.} \end{cases}$$

The ρ_i are well defined thanks to the coherence of (f,α) and (g,β); note that $\rho_0 = 1$. Writing $\boldsymbol{\rho} = (\rho_0,\dots,\rho_{r-1})$, we then find that $\boldsymbol{\rho}^{\mathbf{col}(\boldsymbol{a})} = (\alpha \cup \beta)(\boldsymbol{a})$ for each r-coloured permutation $\boldsymbol{a}$.

Fix $\varepsilon \in \mathbb{Z}$. Let $K\colon \mathbb{Q}[\boldsymbol{p},x][\![t\ast]\!] \to \mathbb{Q}(X)[\![Y\ast]\!]$ be the algebra homomorphism which sends each $\sum_{n=0}^{\infty} h_n(p_0,\dots,p_{r-1},x)t^n$ to $\sum_{n=0}^{\infty} h_n(\rho_0,\dots,\rho_{r-1},X^{\varepsilon})Y^n$. Let $L = K \circ H\colon \mathcal{A}^{(r)}_{(\mathrm{des},\mathrm{comaj},\mathbf{col})} \to \mathbb{Q}(X)[\![Y\ast]\!]$. Write $F = \sum_{\boldsymbol{a}\in\mathrm{supp}(f)} f_{\boldsymbol{a}}[\boldsymbol{a}]$ and $G = \sum_{\boldsymbol{b}\in\mathrm{supp}(g)} g_{\boldsymbol{b}}[\boldsymbol{b}]$. By construction, we then have $L(F) = W^{\varepsilon}_{f,\alpha}$, $L(G) = W^{\varepsilon}_{g,\beta}$, and $L(FG) = W^{\varepsilon}_{f\sqcup g,\alpha\cup\beta}$. The claim follows since

$$W^{\varepsilon}_{f\sqcup g,\alpha\cup\beta} = L(FG) = L(F) \ast_Y L(G) = W^{\varepsilon}_{f,\alpha} \ast_Y W^{\varepsilon}_{g,\beta}.$$ ■

5 Applications to zeta functions

5.1 Ask, class-counting, and orbit-counting zeta functions

The main purpose of the present section is to recall several explicit families of zeta functions associated with algebraic structures (Examples 5.1–5.6). These families will feature in our applications of Theorem 2.2 in Section 5.3. For further details on and motivation for the study of these zeta functions, see [32, 33]. In order to maintain consistency with the literature, we regard $d \times e$ matrices over a ring R as homomorphisms $R^d \to R^e$ acting by right multiplication.

Global ask zeta functions. Given a module $M \subseteq \mathrm{M}_{d\times e}(\mathbb{Z})$ of integral matrices, for each $n \geqslant 1$, let $M_n \subseteq \mathrm{M}_{d\times e}(\mathbb{Z}/n\mathbb{Z})$ denote the reduction of M modulo n. The **(global) ask zeta function** [32, Defn 3.1(i)] of M is the Dirichlet series $\zeta^{\mathrm{ask}}_M(s) = \sum_{n=1}^{\infty} a_n(M)n^{-s}$, where $a_n(M) \in \mathbb{Q}$ denotes the average size of the kernel of matrices in M_n. By the Chinese remainder theorem, $\zeta^{\mathrm{ask}}_M(s) = \prod_p \zeta^{\mathrm{ask}}_{M_p}(s)$ (Euler product), where the product is taken over all primes p and the **local factor** at p is given by $\zeta^{\mathrm{ask}}_{M_p}(s) = \sum_{k=0}^{\infty} a_{p^k}(M)p^{-ks}$, a power series in p^{-s}; see [32, Prop. 3.4(ii)].

Drawing upon deep results from p-adic integration and the theory of zeta functions of algebraic structures, it is known that each $\zeta^{\mathrm{ask}}_{M_p}(s)$ is rational in p^{-s}; see [32, Thm 1.4]. Moreover, as p varies, $\zeta^{\mathrm{ask}}_{M_p}(s)$ is expressible as a weighted sum of rational functions in p and p^{-s}, where the weights are the numbers of $\mathbb{F}_p$-rational points on certain $\mathbb{Z}$-defined varieties; see [32, Thm 4.11]. Thus, these zeta functions generally exhibit delicate arithmetic features. In the present article, we will exclusively focus on so-called uniform examples. These are given by modules of matrices M for which there exists a single bivariate rational function $W(X,Y) \in \mathbb{Q}(X,Y)$ such that $\zeta^{\mathrm{ask}}_{M_p}(s) = W(p, p^{-s})$ for all primes p, perhaps allowing for finitely many exceptions.

Local ask zeta functions. It is often advantageous to bypass global structures altogether and directly study variants of the local factors from above. Let $\mathfrak{O}$ be a compact discrete valuation ring. Let $\mathfrak{P}$ be the maximal ideal of $\mathfrak{O}$ and let q denote the size of the residue field $\mathfrak{O}/\mathfrak{P}$. Such rings $\mathfrak{O}$ are precisely the valuation rings of non-Archimedean local fields. Examples include the ring of p-adic integers $\mathbb{Z}_p$ (in which case $\mathfrak{P} = p\mathbb{Z}_p$, $q = p$, and $\mathfrak{O}/\mathfrak{P} \cong \mathbb{F}_p$) and the ring $\mathbb{F}_q[\![z]\!]$ of formal power series over $\mathbb{F}_q$ (in which case $\mathfrak{P} = z\mathbb{F}_q[\![z]\!]$).

Given a module of matrices $M \subseteq \mathrm{M}_{d\times e}(\mathfrak{O})$, its associated **(local) ask zeta function** is the formal power series $\mathsf{Z}^{\mathrm{ask}}_M(Y) = \sum_{k=0}^{\infty} \alpha_k(M) Y^k$, where $\alpha_k(M)$ denotes the average size of the kernels within the reduction of M modulo $\mathfrak{P}^k$. Local ask zeta functions generalise local factors of global ask zeta functions. Indeed, for a submodule $M \subseteq \mathrm{M}_{d\times e}(\mathbb{Z})$ and prime p, let M_p denote the $\mathbb{Z}_p$-submodule of $\mathrm{M}_{d\times e}(\mathbb{Z}_p)$ generated by M. Then $\mathsf{Z}^{\mathrm{ask}}_{M_p}(p^{-s})$ coincides with the local factor $\zeta^{\mathrm{ask}}_{M_p}(s)$ as defined above.

Explicit formulae for ask zeta functions associated with various "well-known" families of modules of matrices are recorded in [32, §5]. The following examples are of particular interest to us here.

Example 5.1. $\mathsf{Z}^{\mathrm{ask}}_{\mathrm{M}_{d\times e}(\mathfrak{O})}(Y) = \frac{1-q^{-e}Y}{(1-Y)(1-q^{d-e}Y)}$; see [32, Prop. 1.5].

Example 5.2. Let $\mathfrak{O}$ have characteristic distinct from 2. Let $\mathfrak{so}_d(\mathfrak{O})$ be the module of antisymmetric $d \times d$ matrices over $\mathfrak{O}$. By [32, Prop. 5.11], $\mathsf{Z}^{\mathrm{ask}}_{\mathfrak{so}_d(\mathfrak{O})}(Y) = \frac{1-q^{1-d}Y}{(1-Y)(1-qY)} = \mathsf{Z}^{\mathrm{ask}}_{\mathrm{M}_{d\times(d-1)}(\mathfrak{O})}(Y)$.

Example 5.3. Let $\mathfrak{n}_d(\mathfrak{O})$ be the module of strictly upper triangular $d \times d$ matrices over $\mathfrak{O}$. By [32, Prop. 5.15(i)], $\mathsf{Z}^{\mathrm{ask}}_{\mathfrak{n}_d(\mathfrak{O})}(Y) = \frac{(1-Y)^{d-1}}{(1-qY)^d}$.

Class- and orbit-counting zeta functions. Let $\mathfrak{O}$ be a compact discrete valuation ring as above. Let G be a linear group scheme over $\mathfrak{O}$, with a given embedding into $d \times d$ matrices.

The **orbit-counting zeta function** of G is the generating function $\mathsf{Z}^{\mathrm{oc}}_{\mathsf{G}}(Y) = \sum_{k=0}^{\infty} b_k(\mathsf{G}) Y^k$, where $b_k(\mathsf{G})$ denotes the number of orbits of the (finite) matrix group $\mathsf{G}(\mathfrak{O}/\mathfrak{P}^k)$ acting by right-multiplication on its natural module $(\mathfrak{O}/\mathfrak{P}^k)^d$. Apart from the given linear action of G, it is also natural to let G act on itself by conjugation. The **class-counting zeta function** of G is the generating function $\mathsf{Z}^{\mathrm{cc}}_{\mathsf{G}}(Y) = \sum_{k=0}^{\infty} c_k(\mathsf{G}) Y^k$, where $c_k(\mathsf{G})$ denotes the number of conjugacy classes of $\mathsf{G}(\mathfrak{O}/\mathfrak{P}^k)$. Class-counting zeta functions go back to work of du Sautoy [18]. As shown in [32,33], if G is unipotent, then, subject to (mild) restrictions on the residue field size q of $\mathfrak{O}$, the class- and orbit-counting zeta functions of G are instances of ask zeta functions associated with modules of matrices over $\mathfrak{O}$. (These modules can be explicitly described in terms of the Lie algebra of G.) When passing between ask and class-counting zeta functions, one often needs to apply a transformation $Y \leftarrow q^m Y$ for a suitable integer m; see below for an example.

Example 5.4. Suppose that the residue field size q of $\mathfrak{O}$ is odd. By exponentiation, the free class-2-nilpotent Lie algebra on d generators over $\mathfrak{O}$ gives rise to a group scheme $\mathsf{F}_{2,d}$ over $\mathfrak{O}$. We may view $\mathsf{F}_{2,d}$ as an analogue of the free class-2-nilpotent group on d generators. Lins [24, Cor. 1.5] showed that

$$\mathsf{Z}^{\mathrm{cc}}_{\mathsf{F}_{2,d}}(Y) = \frac{1 - q^{\binom{d-1}{2}}Y}{\left(1 - q^{\binom{d}{2}}Y\right)\left(1 - q^{\binom{d}{2}+1}Y\right)}.$$

Looking back at Example 5.2, we observe that $\mathsf{Z}^{\mathrm{cc}}_{\mathsf{F}_{2,d}}(Y) = \mathsf{Z}^{\mathrm{ask}}_{\mathfrak{so}_d(\mathfrak{O})}(q^{\binom{d}{2}}Y)$; this is no coincidence, see [33, Ex. 7.3].

Example 5.5. Let U_d be the group scheme of upper unitriangular $d \times d$ matrices over $\mathfrak{O}$. Suppose that $\gcd(q, (d-1)!) = 1$. By [32, Thm 1.7] (cf. [10, Prop. 4.12]) and Example 5.3, we have $\mathsf{Z}^{\mathrm{oc}}_{\mathsf{U}_d}(Y) = \mathsf{Z}^{\mathrm{ask}}_{\mathfrak{n}_d(\mathfrak{O})}(Y) = \frac{(1-Y)^{d-1}}{(1-qY)^d}$.

Graphs and graphical groups. Given a (finite, simple) graph Γ with distinct vertices $v_1, \dots, v_n$. let M_Γ be the $\mathfrak{O}$-module of alternating $n \times n$ matrices $A = [a_{ij}]$ such that $a_{ij} = 0$ whenever v_i and v_j are non-adjacent. (Here, a matrix is alternating if it is antisymmetric and all diagonal entries are zero. Alternating and antisymmetric matrices coincide for $0 \neq 2$.) We write $\mathsf{Z}^{\mathrm{ask}}_\Gamma(Y)$ for $\mathsf{Z}^{\mathrm{ask}}_{M_\Gamma}(Y)$. As shown in [36, Thm A], $\mathsf{Z}^{\mathrm{ask}}_\Gamma(Y)$ is a rational function in q and Y, without any restrictions on q. In [36, §3.4], the graphical group scheme G_Γ associated with Γ is constructed; for an alternative but equivalent construction, see [34, §1.1]. By [36, Prop. 3.9], $\mathsf{Z}^{\mathrm{cc}}_{\mathsf{G}_\Gamma}(Y) = \mathsf{Z}^{\mathrm{ask}}_\Gamma(q^m Y)$, where m is the number of edges of Γ. Given graphs Γ_1 and Γ_2, let $\Gamma_1 \vee \Gamma_2$ denote their join, obtained from the disjoint union $\Gamma_1 \oplus \Gamma_2$ of Γ_1 and Γ_2 by adding edges connecting each vertex of Γ_1 to each vertex of Γ_2. Let K_n (resp. Δ_n) denote the complete (resp. edgeless) graph on n vertices.

Example 5.6. The graph $\Delta_n \vee \mathsf{K}_{n+1}$ is the threshold graph $\mathrm{Thr}(n, n+1)$ in the notation from [36, §8.4]; this graph has $3\binom{n+1}{2}$ edges. It follows from [36, Thm 8.18] that

$$\mathsf{Z}^{\mathrm{ask}}_{\Delta_n \vee \mathsf{K}_{n+1}}(Y) = \frac{(1 - q^{-n}Y)(1 - q^{-n-1}Y)}{(1 - q^{-1}Y)(1 - Y)(1 - qY)}.$$

Example 5.7. In the same spirit, $\mathsf{T}_n := ((\Delta_n \vee \mathsf{K}_{n+1}) \oplus \Delta_{n+2}) \vee \mathsf{K}_{n+4}$ is the threshold graph $\mathrm{Thr}(n, n+1, n+2, n+4)$. This graph has $3n^2 + 12n + 18 = \binom{n+4}{2} + 3\binom{n+1}{2} + (n+4)(3n+3)$ edges. By [36, Thm 8.18],

$$\mathsf{Z}^{\mathrm{ask}}_{\mathsf{T}_n}(Y) = \frac{(1 - q^{-n-4}Y)(1 - q^{-n-3}Y)^2(1 - q^{-n-2}Y)}{(1 - q^{-3}Y)(1 - q^{-2}Y)(1 - q^{-1}Y)(1 - Y)(1 - qY)}.$$

Hadamard products and zeta functions. Let $\mathfrak{O}$ be as above. Elaborating further on what we wrote in the introduction, modules of matrices, (linear) group schemes, and graphs all admit natural operations which correspond to taking Hadamard products of associated zeta functions. In detail, given modules $M \subseteq \mathrm{M}_{d\times e}(\mathfrak{O})$ and $M' \subseteq \mathrm{M}_{d'\times e'}(\mathfrak{O})$, we regard $M \oplus M'$ as a submodule of $\mathrm{M}_{(d+d')\times(e+e')}(\mathfrak{O})$, embedded diagonally—we will refer to this as a **diagonal direct sum**. Then $\mathsf{Z}^{\mathrm{ask}}_{M\oplus M'}(Y) = \mathsf{Z}^{\mathrm{ask}}_M(Y) *_Y \mathsf{Z}^{\mathrm{ask}}_{M'}(Y)$. Similarly, given (linear) group schemes G and G' over $\mathfrak{O}$, embedded into $d \times d$ and $d' \times d'$ matrices, respectively, we embed $\mathsf{G} \times \mathsf{G}'$ diagonally into $(d+d') \times (d+d')$ matrices. We have $\mathsf{Z}^{\mathrm{oc}}_{\mathsf{G}\times\mathsf{G}'}(Y) = \mathsf{Z}^{\mathrm{oc}}_{\mathsf{G}}(Y) *_Y \mathsf{Z}^{\mathrm{oc}}_{\mathsf{G}'}(Y)$ and $\mathsf{Z}^{\mathrm{cc}}_{\mathsf{G}\times\mathsf{G}'}(Y) = \mathsf{Z}^{\mathrm{cc}}_{\mathsf{G}}(Y) *_Y \mathsf{Z}^{\mathrm{cc}}_{\mathsf{G}'}(Y)$. Finally, given graphs $\mathsf{\Gamma}$ and $\mathsf{\Gamma}'$, we have $\mathsf{Z}^{\mathrm{ask}}_{\mathsf{\Gamma}\oplus\mathsf{\Gamma}'}(Y) = \mathsf{Z}^{\mathrm{ask}}_{\mathsf{\Gamma}}(Y) *_Y \mathsf{Z}^{\mathrm{ask}}_{\mathsf{\Gamma}'}(Y)$; moreover, $\mathsf{G}_{\mathsf{\Gamma}\oplus\mathsf{\Gamma}'} \cong \mathsf{G}_{\mathsf{\Gamma}} \times \mathsf{G}_{\mathsf{\Gamma}'}$.

In this way, explicit formulae and algorithms for computing Hadamard products of rational generating functions become applicable to symbolic enumeration problems in algebra.

5.2 Zeta functions from labelled coloured configurations

It turns out that each zeta function from Examples 5.1–5.7 can be expressed in terms of the rational functions $W^{e}_{f,\alpha}(X,Y)$ attached to labelled coloured configurations as in Section 2. In the following table, we write $\underline{n}$ for the sum of all 2^n coloured permutations of the form $1^{\nu_1}\cdots n^{\nu_n}$ with $\nu_i \in \{0,i\}$. A "%" indicates that an entry coincides with the one immediately above it.

Zeta function	f	α	ε	$u(X)$	$W^{\varepsilon}_{f,\alpha}(X,Y)$
$\mathsf{Z}^{\mathrm{ask}}_{\mathrm{M}_{d\times e}(\mathfrak{O})}(Y)$	$\underline{1}$	$1 \leftarrow -X^{-d}$	$d-e$	1	$\frac{1-X^{-e}Y}{(1-Y)(1-X^{d-e}Y)}$
$\mathsf{Z}^{\mathrm{ask}}_{\mathfrak{so}_d(\mathfrak{O})}(Y)$, $\mathsf{Z}^{\mathrm{ask}}_{\mathrm{M}_{d\times(d-1)}(\mathfrak{O})}(Y)$	%	%	1	%	$\frac{1-X^{1-d}Y}{(1-Y)(1-XY)}$
$\mathsf{Z}^{\mathrm{cc}}_{\mathsf{F}_{2,d}}(Y)$	%	%	%	$X^{\binom{d}{2}}$	%
$\mathsf{Z}^{\mathrm{ask}}_{\mathsf{\Delta}_n\vee\mathsf{K}_{n+1}}(Y)$	$\underline{2}$	$1,2 \leftarrow -X^{-n-1}$	1	X^{-1}	$\frac{(1-X^{1-n}Y)(1-X^{-n}Y)}{(1-Y)(1-XY)(1-X^2Y)}$
$\mathsf{Z}^{\mathrm{cc}}_{\mathsf{G}_{\mathsf{\Delta}_n\vee\mathsf{K}_{n+1}}}(Y)$	%	%	%	$X^{3\binom{n+1}{2}-1}$	%
$\mathsf{Z}^{\mathrm{ask}}_{\mathsf{T}_n}(Y)$	$\underline{4}$	$1,2 \leftarrow -X^{-n-3}$ $3,4 \leftarrow -X^{-n-2}$	1	X^{-3}	$\frac{(1-X^{-n-1}Y)(1-X^{-n}Y)^2(1-X^{1-n}Y)}{\prod_{i=0}^{4}(1-X^iY)}$
$\mathsf{Z}^{\mathrm{cc}}_{\mathsf{T}_n}(Y)$	%	%	%	$X^{5(n+3)(n+1)}$	%
$\mathsf{Z}^{\mathrm{oc}}_{\mathsf{U}_{d+1}}(Y)$	$\underline{d}$	$1,\ldots,d \leftarrow -X^{-1}$	0	X	$\frac{(1-X^{-1}Y)^d}{(1-Y)^{d+1}}$

Table 1: Examples of zeta functions from labelled coloured configurations

Proposition 5.8. *Let $\mathfrak{O}$ be a compact discrete valuation ring with residue field size q. Let $\mathsf{Z}(Y)$ be one of the rational generating functions in the first column of Table 1. We make the following additional assumptions:*

- *If $\mathsf{Z}(Y) = \mathsf{Z}^{\mathrm{ask}}_{\mathfrak{so}_d(\mathfrak{O})}(Y)$, then we assume that $\mathfrak{O}$ has characteristic distinct from 2.*
- *If $\mathsf{Z}(Y) = \mathsf{Z}^{\mathrm{cc}}_{\mathsf{F}_{2,d}}(Y)$, then we assume that q is odd.*
- *If $\mathsf{Z}(Y) = \mathsf{Z}^{\mathrm{oc}}_{\mathsf{U}_d}(Y)$, then we assume that $\gcd(q,(d-1)!) = 1$.*

Let f, α, ε, and $u(X)$ be as in the corresponding row of Table 1. Then

$$\mathsf{Z}(Y) = W^{\varepsilon}_{f,\alpha}(q, u(q)Y).$$

Proof. These are straightforward calculations based on the explicit formulae listed above. We verify two cases and leave the others to the reader.

- $\mathsf{Z}(Y) = \mathsf{Z}^{\mathrm{ask}}_{\mathrm{M}_{d\times e}(\mathfrak{O})}(Y)$: in this case, using Example 2.1 and Example 5.1, we find that

$$W^{\varepsilon}_{\underline{1},\alpha}(q,Y) = \left(\frac{1-X^{-d+\varepsilon}Y}{(1-Y)(1-X^{\varepsilon}Y)}\right)(X \leftarrow q) = \mathsf{Z}^{\mathrm{ask}}_{\mathrm{M}_{d\times e}(\mathfrak{O})}(Y).$$

- $\mathsf{Z}(Y) = \mathsf{Z}^{\mathrm{oc}}_{\mathsf{U}_d}(Y)$: by Example 5.5, we have $\mathsf{Z}^{\mathrm{oc}}_{\mathsf{U}_{d+1}}(q^{-1}Y) = \frac{(1-q^{-1}Y)^d}{(1-Y)^{d+1}}$. A coloured permutation $\boldsymbol{a} = 1^{\nu_1}\cdots d^{\nu_d}$ with $\nu_i \in \{0,i\}$ is uniquely determined by its descent set $J = \mathrm{Des}(\boldsymbol{a}) = \{i \in \{0,\dots,d-1\} : \nu_{i+1} \neq 0\}$. For the given function α, we then have $\alpha(\boldsymbol{a}) = (-X)^{-|J|}$. The claim follows since

$$W^0_{\underline{d},\alpha}(X,Y) = \frac{\sum_{J\subseteq\{0,\dots,d-1\}}(-X)^{-|J|}Y^{|J|}}{(1-Y)^{d+1}} = \frac{(1-X^{-1}Y)^d}{(1-Y)^{d+1}}$$

by the Binomial Theorem. ■

5.3 Applications

Let $\mathfrak{O}$ be a compact discrete valuation ring with residue field size q. We now explain how, subject to a compatibility condition, Theorem 2.2 can be used to explicitly compute Hadamard products of the zeta functions in Table 1. As explained in Section 5.1, we can interpret such Hadamard products as zeta functions associated with suitable products of the objects under consideration. We first record an elementary observation.

Lemma 5.9. *Let R be a ring. Let $A(Y) = \sum_{k=0}^{\infty} a_kY^k$ and $B(Y) = \sum_{k=0}^{\infty} b_kY^k$ be formal power series over R. Let $u,v \in R$. Then $A(uY) *_Y B(vY) = (A *_Y B)(uvY)$.* ■

As in Section 2.2, let $\mathbb{U} = \{\pm X^k : k \in \mathbb{Z}\}$. By combining Theorem 2.2 and Lemma 5.9, we obtain the following.

Corollary 5.10. *Let (f,α) and (g,β) be labelled coloured configurations. Let $\varepsilon \in \mathbb{Z}$ and let $u(X), v(X) \in \mathbb{U}$. Suppose that f and g are strongly disjoint. Then*

$$W^{\varepsilon}_{f,\alpha}(X,\, u(X)Y) *_Y W^{\varepsilon}_{g,\beta}(X,\, v(X)Y) = W^{\varepsilon}_{f\sqcup g,\alpha\beta}(X,\, u(X)v(X)Y). \qquad ■$$

As detailed in Section 2 (and explained in terms of $(\mathrm{des},\mathrm{comaj},\mathbf{col})$-equivalence in Section 4), given labelled coloured configurations (f,α) and (g,β), for the purpose of computing $W^{\varepsilon}_{f,\alpha}(X,\, u(X)Y) *_Y W^{\varepsilon}_{g,\beta}(X,\, v(X)Y)$, we may always pass to equivalent labelled coloured configurations such that f and g are strongly disjoint. (We will tacitly do so in the proofs below.) The compatibility condition that we alluded to above is that we require entries in the ε-column of Table 1 to agree for us to compute associated Hadamard products via Corollary 5.10. All that remains to obtain our zeta function is then to specialise $X \leftarrow q$.

We now record what we regard as the most appealing and interesting applications of Theorem 2.2 in the context of Proposition 5.8 and Table 1. First, given a set of (uncoloured) permutations P, we write

$$\Pi(P) = \{\sigma_1^{\gamma_1}\cdots\sigma_n^{\gamma_n} : \sigma_1\cdots\sigma_n \in P,\ \gamma_i \in \{0,\sigma_i\} \text{ for } i = 1,\dots,n\}.$$

As usual, we write $\mathrm{S}_n = \{\sigma_1\cdots\sigma_n : \{\sigma_1,\dots,\sigma_n\} = [n]\}$ for the symmetric group on n points.

For a function $\alpha\colon \Gamma \to \mathbb{U}$ and coloured permutation $\boldsymbol{a}$ as in Section 2.2, we write $\alpha_q(\boldsymbol{a}) = (\alpha(\boldsymbol{a}))(X \leftarrow q)$.

Corollary 5.11. *Let $d_1,\dots,d_n,e_1,\dots,e_n$ be such that $\delta := d_1 - e_1 = \cdots = d_n - e_n$. Let $\alpha\colon \Gamma \to \mathbb{U}$ with $\alpha(i) = -X^{-d_i}$ for $i = 1,\dots,n$ and $\alpha(c) = 1$ otherwise. Then*

$$\mathsf{Z}^{\mathrm{ask}}_{\mathrm{M}_{d_1\times e_1}(\mathfrak{O})\oplus\cdots\oplus\mathrm{M}_{d_n\times e_n}(\mathfrak{O})}(Y) = \frac{\sum_{\boldsymbol{a}\in\Pi(\mathrm{S}_n)} \alpha_q(\boldsymbol{a})q^{\delta\,\mathrm{comaj}(\boldsymbol{a})}Y^{\mathrm{des}(\boldsymbol{a})}}{(1-Y)(1-q^{\delta}Y)\cdots(1-q^{n\delta}Y)}.$$

Proof. Let $\alpha_i\colon \Gamma \to \mathbb{U}$ with $\alpha_i(i) = -X^{-d_i}$ and $\alpha(c) = 1$ otherwise; note that $\alpha = \alpha_1 \cdots \alpha_n$. By Proposition 5.8, we have $\mathsf{Z}^{\mathrm{ask}}_{\mathrm{M}_{d_i\times e_i}(\mathfrak{O})}(Y) = W^{\delta}_{i^0+i^i,\alpha_i}(q,Y)$. By Theorem 2.2, we thus have

$$\mathsf{Z}^{\mathrm{ask}}_{\mathrm{M}_{d_1\times e_1}(\mathfrak{O})\oplus\cdots\oplus\mathrm{M}_{d_n\times e_n}(\mathfrak{O})}(Y) = \mathop{\ast}_{i=1}^{n} \mathsf{Z}^{\mathrm{ask}}_{\mathrm{M}_{d_i\times e_i}(\mathfrak{O})}(Y) = W^{\delta}_{(1^0+1^1)\sqcup\cdots\sqcup(n^0+n^n),\alpha}(q,Y).$$

The claim follows since the iterated shuffles in question are precisely the elements of $\Pi(\mathrm{S}_n)$. ∎

We note that Corollary 5.11 completely answers [36, Question 10.4(a)] and partially answers [36, Question 10.4(b)]. What remains elusive is an explicit formula for the Hadamard products in Corollary 5.11 *without* our assumption that $d_1 - e_1 = \cdots = d_n - e_n$.

Remark 5.12. Given *arbitrary* d_i and e_i, a formula for

$$\mathsf{Z}^{\mathrm{ask}}_{\mathrm{M}_{d_1\times e_1}(\mathfrak{O})\oplus\cdots\oplus\mathrm{M}_{d_n\times e_n}(\mathfrak{O})}(Y) = \mathsf{Z}^{\mathrm{ask}}_{\mathrm{M}_{d_1\times e_1}(\mathfrak{O})} \ast_Y \cdots \ast_Y \mathsf{Z}^{\mathrm{ask}}_{\mathrm{M}_{d_n\times e_n}(\mathfrak{O})}(Y)$$

as a sum of $\Theta\big(\frac{n!}{(\log 2)^n}\big)$ explicit rational functions appears in [36, Cor. 5.15]. It, however, remains unclear how to derive a meaningful expression of these Hadamard products as quotients of explicit combinatorially-defined polynomials. In case $d_1 - e_1 = \cdots = d_n - e_n$, Corollary 5.11 does just that.

Remark 5.13. There is an evident bijection between $\Pi(\mathrm{S}_n)$ and the group B_n of signed permutations on n points (of order $2^n n!$). With this identification, the special case $d_1 = e_1 = \cdots = d_n = e_n$ of Corollary 5.11 (essentially) coincides with [36, Prop. 10.3] and its close relative [32, Cor. 5.17] (which is the case $d_1 = e_1 = \cdots = d_n = e_n = 1$). To our knowledge, prior to the present work, these were the only known examples of Hadamard products of ask (as well as class- and orbit-counting) zeta functions expressed in terms of (coloured) permutation statistics. In the aforementioned results in the literature, the proofs rely on work of Brenti [6]. These proofs are based on the coincidence of the rational generating functions in question with those attached to so-called q-Eulerian polynomials of signed permutations. This preceded more recent machinery surrounding shuffle compatibility. In particular, this earlier work is now explained as part of the framework presented here.

The following is a group-theoretic application of Corollary 5.11.

Corollary 5.14. *Let $d_1,\dots,d_n \geqslant 1$. Define α as in Corollary 5.11. Suppose that q is odd. Then*

$$\mathsf{Z}^{\mathrm{cc}}_{\mathsf{F}_{2,d_1}\times\cdots\times\mathsf{F}_{2,d_n}}\big(q^{-\sum_{i=1}^{n}\binom{d_i}{2}}Y\big) = \frac{\sum_{\boldsymbol{a}\in\Pi(\mathrm{S}_n)} \alpha_q(\boldsymbol{a})q^{\mathrm{comaj}(\boldsymbol{a})}Y^{\mathrm{des}(\boldsymbol{a})}}{(1-Y)(1-qY)\cdots(1-q^nY)}.$$

Proof. This follows since $\mathsf{Z}^{\mathrm{cc}}_{\mathsf{F}_{2,d}}(Y) = \mathsf{Z}^{\mathrm{ask}}_{\mathrm{M}_{d\times(d-1)}(\mathfrak{O})}(q^{\binom{d}{2}}Y)$. ∎

We can also symbolically compute the orbit-counting zeta function of $\mathsf{U}_{d_1+1}\times\cdots\times\mathsf{U}_{d_n+1}$.

Corollary 5.15. *Given $d_1,\dots,d_n \geqslant 0$, write $D_i = d_1 + \cdots + d_i$ and*

$$T = (D_0+1)\cdots D_1 \sqcup \cdots \sqcup (D_{n-1}+1)\cdots D_n \subseteq \mathrm{S}_{D_n},$$

a set of uncoloured permutations of cardinality $\prod_{i=1}^{n}\binom{D_i}{D_{i-1}}$. Recall the definition of pal^* *from Section 2.1. Let $m = \max_{i=1,\dots,n} d_i$. Suppose that* $\gcd(q,(m-1)!) = 1$. *Then*

$$\mathsf{Z}^{\mathrm{oc}}_{\mathsf{U}_{d_1+1}\times\cdots\times\mathsf{U}_{d_n+1}}(q^{-n}Y) = \mathop{\ast}_{i=1}^{n} \mathsf{Z}^{\mathrm{oc}}_{\mathsf{U}_{d_i+1}}(q^{-1}Y) = \frac{\sum_{\boldsymbol{a}\in\Pi(T)}(-q)^{-|\mathrm{pal}^*(\boldsymbol{a})|}\,Y^{\mathrm{des}(\boldsymbol{a})}}{(1-Y)^{D_n+1}}.$$

Proof. Define $\alpha\colon \Gamma \to \mathbb{U}$ with $\alpha(j) = -X^{-1}$ for $j = 1, \ldots, D_n$ and $\alpha(c) = 1$ otherwise. Let $\alpha_i\colon \Gamma \to \mathbb{U}$ with $\alpha_i(j) = -X^{-1}$ for $j = D_{i-1}+1, \ldots, D_i$ and $\alpha_i(c) = 1$ otherwise; note that $\alpha = \alpha_1 \cdots \alpha_n$. By construction, for $\boldsymbol{a} \in \Pi(T)$, we have $\alpha_q(\boldsymbol{a}) = (-q)^{-|\mathrm{pal}^*(\boldsymbol{a})|}$. Extending our previous notation, for $a \leqslant b$, write $\underline{a \ldots b}$ for the sum over all $a^{\nu_a} \cdots b^{\nu_b}$ with ν_i satisfying $\nu_i \in \{0, i\}$. Then

$$\mathop{\ast}_{i=1}^{n} \mathsf{Z}^{\mathrm{oc}}_{\mathsf{U}_{d_i+1}}(q^{-1}Y) = \mathop{\ast}_{i=1}^{n} W^0_{\underline{D_{i-1}+1\ldots D_i},\alpha_i}(q,Y) = W^0_{\underline{D_0+1\ldots D_1}\sqcup\cdots\sqcup\underline{D_{n-1}+1\ldots D_n},\alpha}(q,Y); \tag{5.1}$$

the gcd condition on q is justified by [32, Cor. 8.16] and the fact that $\mathsf{U}_{d_1} \times \cdots \times \mathsf{U}_{d_r}$ has nilpotency class $\max_{i=1,\ldots,n} d_i - 1$. The claim follows since the iterated shuffles in (5.1) are precisely the elements of $\Pi(T)$. ■

Acknowledgements

AC and TR gratefully acknowledge interesting discussions with Christopher Voll. AC was partially supported by the College of Science and Engineering at the University of Galway through a Strategic Research (Millennium) Fund.

References

[1] R. M. Adin, C. A. Athanasiadis, S. Elizalde, and Y. Roichman, *Character formulas and descents for the hyperoctahedral group*, Adv. in Appl. Math. **87** (2017), 128–169.

[2] M. Berman, I. Glazer, and M. Schein, *Pro-isomorphic zeta functions of nilpotent groups and Lie rings under base extension*, Trans. Amer. Math. Soc. **375** (2022), no. 02, 1051–1100.

[3] R. Biagioli and F. Caselli, *Weighted enumerations on projective reflection groups*, Adv. in Appl. Math. **48** (2012), no. 1, 249–268.

[4] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics, vol. 231, Springer, New York, 2005.

[5] A. Björner and M. L. Wachs, *Permutation statistics and linear extensions of posets*, J. Combin. Theory Ser. A **58** (1991), no. 1, 85–114.

[6] F. Brenti, *q-Eulerian polynomials arising from Coxeter groups*, European J. Combin. **15** (1994), no. 5, 417–441.

[7] F. Brenti and A. Carnevale, *Proof of a conjecture of Klopsch-Voll on Weyl groups of type A*, Trans. Amer. Math. Soc. **369** (2017), no. 10, 7531–7547.

[8] A. Carnevale, *On some Euler-Mahonian distributions*, Electron. J. Combin. **24** (2017), no. 3, Paper No. 3.27, 11 pp.

[9] A. Carnevale, V. D. Moustakas, and T. Rossmann, *From coloured permutations to Hadamard products and zeta functions*, To appear in Sém. Lothar. Combin.

[10] A. Carnevale and T. Rossmann, *Linear relations with disjoint supports and average sizes of kernels*, J. Lond. Math. Soc. (2) **106** (2022), no. 3, 1759–1809.

[11] A. Carnevale, M. M. Schein, and C. Voll, *Generalized Igusa functions and ideal growth in nilpotent Lie rings*, Sém. Lothar. Combin. **84B** (2020), Art. 71, 12 pp.

[12] A. Carnevale, M. M. Schein, and C. Voll, *Generalized Igusa functions and ideal growth in nilpotent Lie rings*, Algebra Number Theory **18** (2024), no. 3, 537–582.

[13] A. Carnevale, S. Shechter, and C. Voll, *Enumerating traceless matrices over compact discrete valuation rings*, Israel J. Math. **227** (2018), no. 2, 957–986.

[14] A. Carnevale and E. Tielker, *On Denert's statistic*, Electron. J. Combin. **29** (2022), no. 3, Paper 3.1, 21 pp.

[15] A. Carnevale and C. Voll, *Orbit Dirichlet series and multiset permutations*, Monatsh. Math. **186** (2018), no. 2, 215–233.

[16] M. Denert, *The genus zeta function of hereditary orders in central simple algebras over global fields*, Math. Comp. **54** (1990), no. 189, 449–465.

[17] G. Dorpalen-Barry, J. Maglione, and C. Stump, *The Poincaré-extended ab-index* (2023). arXiv:2301.05904.

[18] M. P. F. du Sautoy, *Counting conjugacy classes*, Bull. London Math. Soc. **37** (2005), no. 1, 37–44.

[19] D. Foata and D. Zeilberger, *Denert's permutation statistic is indeed Euler-Mahonian*, Stud. Appl. Math. **83** (1990), no. 1, 31–59.

[20] I. M. Gessel and Y. Zhuang, *Shuffle-compatible permutation statistics*, Adv. Math. **332** (2018), 85–141.

[21] D. Grinberg, *Shuffle-compatible permutation statistics II: the exterior peak set*, Electron. J. Combin. **25** (2018), no. 4, Paper No. 4.17, 61.

[22] F. J. Grunewald, D. Segal, and G. C. Smith, *Subgroups of finite index in nilpotent groups*, Invent. Math. **93** (1988), 185–223.

[23] S. K. Hsiao and T. K. Petersen, *Colored posets and colored quasisymmetric functions*, Ann. Comb. **14** (2010), no. 2, 251–289.

[24] P. M. Lins de Araujo, *Bivariate representation and conjugacy class zeta functions associated to unipotent group schemes, II: Groups of type F, G, and H*, Internat. J. Algebra Comput. **30** (2020), no. 5, 931–975.

[25] V. D. Moustakas, *Enumerative combinatorics, representations and quasisymmetric functions*, Ph.D. Thesis, 2021. `http://hdl.handle.net/10442/hedi/51071`.

[26] V.-D. Moustakas, *Specializations of colored quasisymmetric functions*, Sém. Lothar. Combin. **85B** (2021), Art. 49, 12.

[27] V.-D. Moustakas, *Specializations of colored quasisymmetric functions and Euler-Mahonian identities*, Adv. in Appl. Math. **127** (2021), Paper No. 102177, 39 pp.

[28] T. K. Petersen, *Eulerian numbers*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser/Springer, New York, 2015. With a foreword by Richard Stanley.

[29] S. Poirier, *Cycle type and descent set in wreath products*, Proceedings of the 7th Conference on Formal Power Series and Algebraic Combinatorics, 1998, pp. 315–343.

[30] V. Reiner, *Signed permutation statistics*, European J. Combin. **14** (1993), no. 6, 553–567.

[31] T. Rossmann, *A framework for computing zeta functions of groups, algebras, and modules*, Algorithmic and experimental methods in algebra, geometry, and number theory, 2017, pp. 561–586.

[32] T. Rossmann, *The average size of the kernel of a matrix and orbits of linear groups*, Proc. Lond. Math. Soc. (3) **117** (2018), no. 3, 574–616.

[33] T. Rossmann, *The average size of the kernel of a matrix and orbits of linear groups, II: duality*, J. Pure Appl. Algebra **224** (2020), no. 4, Art. 106203, 28 pp.

[34] T. Rossmann, *Enumerating conjugacy classes of graphical groups over finite fields*, Bull. Lond. Math. Soc. **54** (2022), no. 5, 1923–1943.

[35] T. Rossmann, Zeta*, version 0.4.2*, 2022. See `https://github.com/torossmann/Zeta`.

[36] T. Rossmann and C. Voll, *Groups, graphs, and hypergraphs: Average sizes of kernels of generic matrices with support constraints*, Mem. Amer. Math. Soc. **294** (2024), no. 1465, vi+120.

[37] L. Solomon, *Zeta functions and integral representation theory*, Advances in Math. **26** (1977), no. 3, 306–326.

[38] R. P. Stanley, *Enumerative combinatorics*, 2nd ed., Cambridge Studies in Advanced Mathematics, 49, vol. 1, Cambridge University Press, 1997.

[39] A. Stasinski and C. Voll, *A new statistic on the hyperoctahedral groups*, Electron. J. Combin. **20** (2013), no. 3, Paper 50, 23 pp.

[40] A. Stasinski and C. Voll, *Representation zeta functions of nilpotent groups and generating functions for Weyl groups of type B*, Amer. J. Math. **136** (2014), no. 2, 501–550.

[41] E. Steingrímsson, *Permutation statistics of indexed permutations*, European J. Combin. **15** (1994), no. 2, 187–205.

[42] C. Voll, *A newcomer's guide to zeta functions of groups and rings*, Lectures on profinite topics in group theory, 2011, pp. 99–144.