ON DENERT'S STATISTIC

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Abstract. We show that the numerators of genus zeta function associated with local hereditary orders studied by Denert can be described in terms of the joint distribution of Euler-Mahonian statistics on multiset permutations defined by Han. We use this result to deduce a reciprocity property for genus zeta functions of local hereditary orders whose associated composition is a rectangle. We also record a remarkable identity satisfied by genus zeta functions of local hereditary orders in terms of Hadamard products of genus zeta functions of maximal orders. Finally, we define Mahonian companions of excedance statistics on groups of signed and even-signed permutations.

1. Introduction

Recently, permutation statistics have found applications to various zeta functions in algebra; see, for instance, [2, 5, 7, 20, 21]. An early instance of such applications arose from the enumeration of ideals in hereditary orders encoded in so-called genus zeta functions. It is known that local hereditary orders are parameterised by local invariants, which are integer compositions. In order to give an explicit expression for the numerators of genus zeta functions of such orders, Denert [9] defined a pair of statistics over permutations.

Remarkably, for "minimal" (i.e. associated with the all-one composition) hereditary orders, the numerators of the associated genus zeta functions are, for a suitable choice of variables, Euler-Mahonian polynomials over symmetric groups. This was first conjectured by Denert in [9] and then proved by Foata and Zeilberger in [11].

Inspired by Denert's paper, Han [14] gave a definition of a *Denert statistic* for multiset permutations, which together with the classical excedance statistic is Euler-Mahonian. While Han's result provides a Mahonian companion for the excedance statistic already considered by MacMahon [16] on multiset permutations, it does not, to the best of our knowledge, provide a combinatorial interpretation of the numerators of Denert's genus zeta functions; cf. [14, p. 25].

This paper is devoted to a further study of Denert's statistic. In the first part, we close the circle by showing that Denert's pair of statistics (as originally defined) is indeed equidistributed with the Euler-Mahonian statistics considered by Han on multiset permutations; cf. Theorem 4.1. This gives an explicit description of the numerators of the genus zeta functions of local hereditary orders with arbitrary local invariants.

By results going back to MacMahon, our equidistribution result also implies a remarkable identity involving Hadamard products of genus zeta functions of local hereditary orders. Similar identities, also involving Eulerian or Euler-Mahonian polynomials, have appeared in recent work on so-called ask zeta functions [18, 19] and zeta functions associated with quiver representations [15].

Generalisations of Euler-Mahonian identities to signed and even-signed permutations have been extensively studied (see, e.g., [1, 3, 6]). The remainder of this paper is devoted to generalisations of Denert's statistic which provide Mahonian companions to suitable excedance statistics on Coxeter groups of type B and D.

The paper is organised as follows. In Section 2 we collect some notation and preliminaries on permutation statistics on multiset permutations, while in Section 3 we recall Denert's definitions of the statistics appearing in the numerators of the genus zeta functions studied in [9]. Section 4 is devoted to proving that these numerators are indeed Euler-Mahonian polynomials. In Section 5, we define analogues of Denert's statistics in types B and D. Together with suitable excedance statistics, these are equidistributed with Euler-Mahonian statistics on groups of signed and even-signed permutations, respectively. We conclude the paper with a few remarks in Section 6, including the aforementioned identity involving Hadamard products satisfied by Denert's genus zeta functions.

2. Notation and preliminaries

We set $[n] = \{1, \ldots, n\}$ and denote by $\{i_1, \ldots, i_m\}_<$ a set of increasing integers $i_1 < \cdots < i_m$. We let $|S|$ denote the cardinality of a set S. For the remainder of this paper, $\eta = (\eta_1, \ldots, \eta_r)$ is a fixed composition of $n \in \mathbb{N}$ with r parts. Given η, we let S_η denote the set of all permutations of the multiset

$$\{\underbrace{1, \ldots, 1}_{\eta_1}, \ldots, \underbrace{r, \ldots, r}_{\eta_r}\}$$

comprising η_1 copies of 1, η_2 copies of 2, and so on. In other words, a *multiset permutation* in S_η is a rearrangement of the "trivial" word $\mathrm{id}^\eta = 1^{\eta_1} \cdots r^{\eta_r} \in S_\eta$. Note that when $\eta = (1, 1, \ldots, 1)$, S_η is the symmetric group S_r. We will be interested in several statistics on multiset permutations. We denote the *descent set* of $w = w_1 \cdots w_n \in S_\eta$ by

$$\mathrm{Des}(w) = \{i \in [n-1] : w_i > w_{i+1}\}.$$

The *descent* and *major index* statistics are

$$\mathrm{des}(w) = |\,\mathrm{Des}(w)| \quad \text{and} \quad \mathrm{maj}(w) = \sum_{i \in \mathrm{Des}(w)} i.$$

Further, we define the descent set of a composition $\mathrm{Des}(\eta) := \{\eta_1, \eta_1 + \eta_2, \ldots, \sum_{i=1}^{r-1} \eta_i\}$.

In the following, we recall a few definitions in order to define the pair of statistics $(\mathrm{den}, \mathrm{exc})$, see also [14]. When η is fixed, we will simply denote with id the trivial word id^η of the corresponding set of multiset permutations.

A position $i \in [n]$ is an *excedance* of $w \in S_\eta$ if the i-th letter of w is strictly greater than the i-th letter of the trivial word id. We denote with $\mathrm{Exc}(w)$ the set of all excedances of w and with $\mathrm{exc}(w)$ its cardinality, viz.

$$\mathrm{Exc}(w) = \{i \in [n] : w_i > \mathrm{id}_i\} \quad \text{and} \quad \mathrm{exc}(w) = |\,\mathrm{Exc}(w)|. \tag{2.1}$$

Definition 2.1. Let $w \in S_\eta$. The *exceeding subword* of w is

$$\mathbf{exc}(w) := w_{i_1} \cdots w_{i_k} \text{ for } \mathrm{Exc}(w) = \{i_1, \ldots, i_k\}_<.$$

The *non-exceeding* subword of w is

$$\mathbf{nexc}(w) := w_{j_1} \cdots w_{j_{n-k}} \text{ for } \{j_1, \ldots, j_{n-k}\}_< := [n] \backslash \mathrm{Exc}(w).$$

For example, for $\eta = (3, 2, 2, 3)$ and $w = 4232314141$, the exceeding subword is $\mathbf{exc}(w) = 42334$ and the non-exceeding subword is $\mathbf{nexc}(w) = 21141$.

As usual, we let $\mathrm{inv}(w)$ denote the *inversion number* of a multiset permutation $w \in S_\eta$

$$\mathrm{inv}(w) = |\{(i, j) : 1 \leqslant i < j \leqslant n,\, w_i > w_j\}|$$

and $\mathrm{imv}(w)$ denote the *weak inversion number* of w

$$\mathrm{imv}(w) = |\{(i, j) : 1 \leqslant i < j \leqslant n,\, w_i \geqslant w_j\}|.$$

Generalising work of Foata and Zeilberger on permutations [11], Han gave the following definition of a Denert statistic on multiset permutations.

Definition 2.2 ([14, Définition 1.3])**.** Let $w \in S_\eta$. Denert's statistic on multiset permutations is given by

$$\mathrm{den}(w) := \sum_{i \in \mathrm{Exc}(w)} i + \mathrm{imv}(\mathbf{exc}(w)) + \mathrm{inv}(\mathbf{nexc}(w)).$$

For instance, $\mathrm{den}(4232314141) = 18 + 5 + 4 = 27$. Han proved that this statistic, together with the excedance number defined in (2.1), is equidistributed with the pair of statistics $(\mathrm{maj}, \mathrm{des})$ on multiset permutations.

Theorem 2.3 ([14, Théorème 1.4])**.** *The pair of statistics on* S_η $(\mathrm{den}, \mathrm{exc})$ *is Euler-Mahonian, i.e.*

$$\sum_{w \in S_\eta} x^{\mathrm{den}(w)} y^{\mathrm{exc}(w)} = \sum_{w \in S_\eta} x^{\mathrm{maj}(w)} y^{\mathrm{des}(w)}.$$

3. Denert's statistic

Our first main result shows that the polynomials expressing the numerators of the genus zeta functions of hereditary orders with local invariants η and r coincide with the polynomials giving the joint distribution of $(\mathrm{den}, \mathrm{exc})$ over S_η in Theorem 2.3. These numerators, as defined by Denert in [9, Theorem 11], involve statistics on so-called η-admissible permutations, iden and iexc, which we now define, closely following [9].

Let $\sigma \in S_n$. Following Denert, we visualise σ as the matrix whose (i, j)-th entry is defined as

$$(i, j) = \begin{cases} 1 & \text{if } j = \sigma(i), \\ 0 & \text{otherwise.} \end{cases}$$

Note that this is the transpose of the usual permutation matrix associated with σ. Nevertheless, to ease the translation between Denert's and our notation, we will refer to it as the matrix associated with σ. Since we are interested in statistics counting certain zero entries, we think of this matrix as an $n \times n$ grid, and we refer to matrix entries as cells in this grid.

Definition 3.1. The *projection* or *block-map* with respect to the composition η is the map $\pi_\eta : [n] \to [r]$ such that

$$\sum_{k=1}^{\pi_\eta(i)-1} \eta_i < i \leqslant \sum_{k=1}^{\pi_\eta(i)} \eta_i.$$

That is, $\pi_\eta(i) = 1$ for $1 \leqslant i \leqslant \eta_1$, $\pi_\eta(i) = 2$ for $\eta_1 + 1 \leqslant i \leqslant \eta_1 + \eta_2$ and so on.

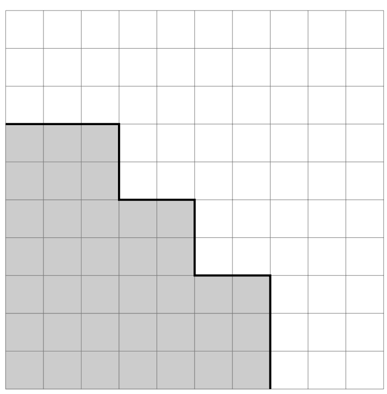

Figure 1. For $n = 10$ and $\eta = (3, 2, 2, 3)$, the set $[>]$ is coloured in grey, while the set $[\preceq]$ is left blank.

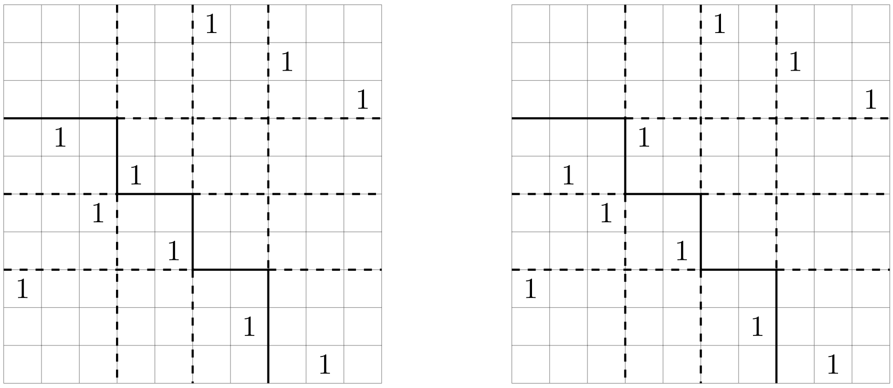

FIGURE 2. Let $n = 10$ and $\eta = (3, 2, 2, 3)$. The left matrix corresponds to $\sigma = 68102435179$ and the right matrix to $\tau = 68104235179$

By slight abuse of notation, we also denote by $\pi_\eta \colon S_n \to S_\eta$ the projection from permutations to multiset permutations

$$\pi_\eta(\sigma) := \pi_\eta(\sigma(1)) \cdots \pi_\eta(\sigma(n)).$$

For instance, $\pi_{(3,2,2,3)}(68102435179) = 3441212134$.

The block-map partitions a permutation matrix into r^2 blocks of size $\eta_i \times \eta_j$, $1 \leqslant i, j \leqslant r$. For $k \in \mathbb{N}$ we define the *k-th block-row* (resp. *k-th block-column*) to be the set of pairs $(i, j) \in [n]^2$ such that $\pi_\eta(i) = k$ (resp. $\pi_\eta(j) = k$). Let further

$$[\preceq] = \{(i, j) : \pi_\eta(i) \leqslant \pi_\eta(j)\},$$
$$[\prec] = \{(i, j) : \pi_\eta(i) < \pi_\eta(j)\},$$
$$[\succ] = \{(i, j) : \pi_\eta(i) > \pi_\eta(j)\}.$$

We illustrate the sets $[\preceq]$ and $[\succ]$ in Figure 1, see also [9, Section 1]. Following Denert, we say that a permutation $\sigma \in S_n$ is *descending on* $I \subseteq [n]^2$ if for all $(i, \sigma(i)), (j, \sigma(j)) \in I$, $i < j$ if and only if $\sigma(i) < \sigma(j)$. For instance, $\sigma = 68102435179 \in S_{(3,2,2,3)}$ is descending on every block-row, but not on the first and last block-column, which can be easily seen in Figure 2.

The polynomials we are interested in are generating polynomials on permutations which Denert calls *η-admissible permutations*. These are permutations whose descent sets are contained in the descent set of the composition η.

Definition 3.2. A permutation $\sigma \in S_n$ is *η-admissible* if it is descending on every block-row. We will denote $S^\eta = \{\sigma \in S_n : \mathrm{Des}(\sigma) \subset \mathrm{Des}(\eta)\}$ the set of all η-admissible permutation in S_n.

For instance, $\sigma = 68102435179$ is $(3, 2, 2, 3)$-admissible, while $\tau = 68104235179$ is not (see also Figure 2). Note that the set of η-admissible permutations is a parabolic quotient of S_n; see, e.g., [4, Section 2.4].

It is well known that parabolic quotients and thus η-admissible permutations are in bijection with the set of multiset permutations S_η via the map $\sigma \mapsto \pi_\eta(\sigma^{-1})$. Indeed, the projection π_η is injective on the set of permutations whose inverses have descent sets contained in $\mathrm{Des}(\eta)$. The inverse of this map is defined in terms of the *standardisation* $\mathrm{std} = \mathrm{std}_\eta : S_\eta \to S_n$. Informally, the standardisation of $w \in S_\eta$ is a permutation $\mathrm{std}(w)$ which we obtain from w by substituting the η_1 1s from left to right with $1, \ldots, \eta_1$, the η_2 2s from left to right with $\eta_1 + 1, \ldots, \eta_1 + \eta_2$ and so on; see also, e.g., [6, Section 2].

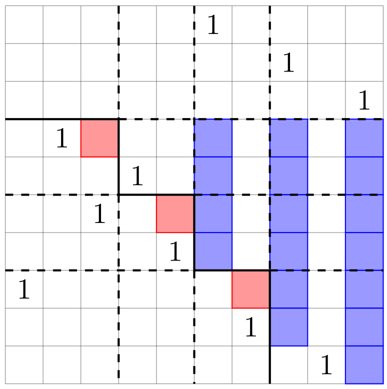

FIGURE 3. Let $n = 10$, $\eta = (3, 2, 2, 3)$ and $\sigma = 68102435179$. Elements of N_σ^+ are marked in blue and elements of N_σ^- are marked in red.

We then obtain an η-admissible permutation by taking the inverse of $\mathrm{std}(w)$. That is,

$$\begin{aligned} S^\eta &\overset{1-1}{\longleftrightarrow} S_\eta \\ \sigma &\mapsto \pi_\eta(\sigma^{-1}) \\ (\mathrm{std}_\eta(w))^{-1} &\leftarrow\!\shortmid w. \end{aligned} \tag{3.1}$$

For instance, for $\eta = (3, 2, 2, 3)$ and $\sigma = 68102435179$, we have $\sigma^{-1} = 84657192103$ and thus $\pi_\eta(\sigma^{-1}) = 4232314141$. On the other hand, $\mathrm{std}(4232314141) = 84657192103 = \sigma^{-1}$, and therefore $(\mathrm{std}(4232314141))^{-1} = \sigma$, as claimed.

This bijection is a key ingredient in the proof of Theorem 4.1. We are now ready to introduce the first of the two statistics needed to show our main result.

Definition 3.3. For $\sigma \in S_n$ and η a composition of n we define

$$I_\sigma = \{(i, \sigma(i)) \in [\succ]\} = \{j : \pi_\eta(\sigma^{-1}(j)) > \pi_\eta(j)\}.$$

Note that I_σ coincides with the set of excedances of $\pi_\eta(\sigma^{-1}))$, that is $I_\sigma = \mathrm{Exc}(\pi_\eta(\sigma^{-1}))$. Therefore, we denote its cardinality with

$$\mathrm{iexc}(\sigma) := |I_\sigma|.$$

Remark 3.4. The statistic iexc appears as k in [9].

Further, we give here the definitions of the sets N_σ^+ and N_σ^-,

$$N_\sigma^+ = [\preceq] \cap \{(i, j) : \sigma(i) < j \text{ and } \sigma^{-1}(j) < i\},$$

and

$$N_\sigma^- = [\succ] \cap \{(i, j) : \sigma(i) < j \text{ and } \sigma^{-1}(j) > i\}.$$

Note that Denert uses the same notation for the cardinalities of these sets; cf. [9, Section 2]. Figure 3 illustrates N_σ^+ and N_σ^- for a permutation in $S_{(3,2,2,3)}$, where we marked elements of N_σ^+ and N_σ^- as coloured cells in the permutation matrix of σ.

The statistic introduced in the next definition implicitly appeared in the numerators of Denert's genus zeta functions. For this reason, we refer to it as Denert's statistic (see also Proposition 3.6).

Definition 3.5. For $\sigma \in S^\eta$, Denert's statistic is defined as

$$\mathrm{iden}(\sigma) = \sum_{j \in I_\sigma} j + |N_\sigma^+| - |N_\sigma^-| - \mathrm{iexc}(\sigma).$$

For instance, for $\sigma = 68102435179$, $\mathrm{iden}(\sigma) = 18 + 17 - 3 - 5 = 27$, see also Figure 3.

Note that thanks to the map (3.1) $\sigma \mapsto \pi_\eta(\sigma^{-1})$, we obtain a statistic on the set of multiset permutations. Our goal is to show that the statistic obtained in this way is indeed Han's statistic from Definition 2.2, which justifies our notation.

As mentioned above, Denert's statistic appears in the numerators of genus zeta functions of local hereditary orders. In the next subsection, we recall the definition of such zeta functions and the main result of [9].

3.1. **Genus zeta functions of local hereditary orders.** For a composition η of n, set

$$W_\eta(x,y) := \frac{\sum_{\sigma\in S^\eta} x^{\mathrm{iden}(\sigma)} y^{\mathrm{iexc}(\sigma)}}{\prod_{0\leqslant j\leqslant n-1}(1-x^i y)} \in \mathbb{Q}(x,y).$$

Then [9, Theorem 11] is a closed formula for the genus zeta function of a local hereditary order in terms of the rational functions W_η.

We briefly recall here the relevant definitions, the aforementioned result and a sketch of its proof.

Let K be a local field and R be its ring of integers. Let A be a central simple algebra over K. Then A is isomorphic to $M_n(D)$ for a unique integer n and division K-algebra D. Let Δ be the unique maximal order in D and let $\mathfrak{p}$ be the unique maximal two-sided ideal of Δ. Write $q = |\Delta/\mathfrak{p}|$.

Given an R-order Θ in A, the genus zeta function of Θ is the Dirichlet series $Z_\Theta(s) = \sum |\Theta : \mathcal{L}|^{-s}$, where the sum ranges over integral free ideals of Θ; cf. [9, Definition 3.1]. It is known that hereditary orders in A are parameterised by so-called local invariants, which are compositions of n. Given any such composition η, an explicit description of a hereditary order Θ^η with local invariant parameterised by an integer composition η can be found in [9, Theorem 7].

Proposition 3.6. $Z_{\Theta^\eta}(s) = W_\eta(q, q^{-ns})$.

(Sketch of) Proof of Proposition 3.6. Following Denert's proof of [9, Theorem 11], we have

$$Z_{\Theta^\eta}(s) = \sum_{\sigma\in S^\eta} q^{|N_\sigma^+|-|N_\sigma^-|} \sum_{\substack{\lambda\in\mathbb{N}^n\\ \lambda_j>0 \text{ if } j\in I_\sigma}} \prod_{1\leqslant j\leqslant n} (q^{j-1-ns})^{\lambda_j}.$$

Setting $t := q^{-ns}$, with an inclusion-exclusion argument we obtain

$$\begin{aligned}
\sum_{\substack{\lambda\in\mathbb{N}^n\\ \lambda_j>0 \text{ if } j\in I_\sigma}} \prod_{1\leqslant j\leqslant n} (q^{j-1}t)^{\lambda_j} &= \sum_{\lambda\in\mathbb{N}^n} \prod_{1\leqslant j\leqslant n} (q^{j-1}t)^{\lambda_j} - \sum_{j\in I_\sigma} \sum_{\substack{\lambda\in\mathbb{N}^n\\ \lambda_j=0}} \prod_{1\leqslant j\leqslant n} (q^{j-1}t)^{\lambda_j}\\
&\quad + \sum_{\{j_1,j_2\}_<\subset I_\sigma} \sum_{\substack{\lambda\in\mathbb{N}^n\\ \lambda_{j_1}=\lambda_{j_2}=0}} \prod_{1\leqslant j\leqslant n} (q^{j-1}t)^{\lambda_j} - \dots (-1)^{|I_\sigma|} \sum_{\substack{\lambda\in\mathbb{N}^n\\ \lambda_j=0 \text{ if } j\in I_\sigma}} \prod_{1\leqslant j\leqslant n} (q^{j-1}t)^{\lambda_j}\\
&= \left(\prod_{1\leqslant j\leqslant n} (1-q^{j-1}t)\right)^{-1} \left(1 + \sum_{\varnothing\neq J\subseteq I_\sigma} (-1)^{|J|} \prod_{j\in J} (1-q^{j-1}t)\right)\\
&= \left(\prod_{1\leqslant j\leqslant n} (1-q^{j-1}t)\right)^{-1} \prod_{j\in I_\sigma} q^{j-1}t.
\end{aligned}$$

Therefore,

$$\begin{aligned} Z_{\Theta^\eta}(s) &= \frac{\sum_{\sigma\in S^\eta} q^{|N_\sigma^+|-|N_\sigma^-|}\prod_{j\in I_\sigma} q^{j-1-ns|I_\sigma|}}{\prod_{1\leqslant j\leqslant n}(1-q^{j-1-ns})} \\ &= \frac{\sum_{\sigma\in S^\eta} q^{|N_\sigma^+|-|N_\sigma^-|+\sum_{j\in I_\sigma} j-(1+ns)\,\mathrm{iexc}(\sigma)}}{\prod_{1\leqslant j\leqslant n}(1-q^{j-1-ns})} \\ &= \frac{\sum_{\sigma\in S^\eta} q^{\mathrm{iden}(\sigma)-ns\,\mathrm{iexc}(\sigma)}}{\prod_{0\leqslant i\leqslant n-1}(1-q^{i-ns})}, \end{aligned}$$

as claimed. □

4. Denert's genus zeta function and Euler-Mahonian polynomials

In this section we prove our theorem about the equidistribution of (den, exc) over the set of multiset permutations S_η and that of (iden, iexc) over the set of η-admissible permutations S^η.

Theorem 4.1. *The pair of statistics* (iden, iexc) *is Euler-Mahonian, i.e.*

$$\sum_{\sigma\in S^\eta} x^{\mathrm{iden}(\sigma)} y^{\mathrm{iexc}(\sigma)} = \sum_{w\in S_\eta} x^{\mathrm{den}(w)} y^{\mathrm{exc}(w)}.$$

In preparation for the proof, we further partition the set N_σ^+ into

$$N_\sigma^+[\preceq] = \{(i,j) : \sigma(i) < j, \sigma^{-1}(j) < i, \pi_\eta(i) \leqslant \pi_\eta(j), \underbrace{\pi_\eta(i) \leqslant \pi_\eta(\sigma(i))}_{\text{i.e. } (i,\sigma(i))\in[\preceq]}\}$$

and

$$N_\sigma^+[\succ] = \{(i,j) : \sigma(i) < j, \sigma^{-1}(j) < i, \pi_\eta(i) \leqslant \pi_\eta(j), \underbrace{\pi_\eta(i) > \pi_\eta(\sigma(i))}_{\text{i.e. } (i,\sigma(i))\in[\succ]}\},$$

see Figure 4 for an example.

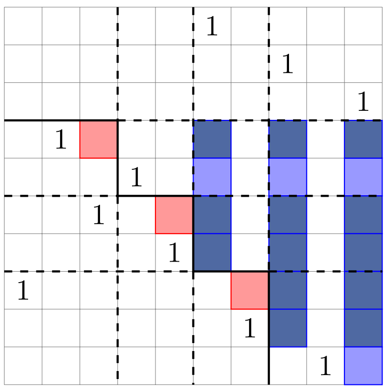

Figure 4. Let $n = 10$ and $\eta = (3,2,2,3)$. For $\sigma = 68102435179$ the set $N_\sigma^+[\preceq]$ is marked in blue, the set $N_\sigma^+[\succ]$ is marked in dark blue, while the set N_σ^- is marked in red.

The following technical lemmata are key to show that $\mathrm{iden}(\sigma) = \mathrm{den}(\pi_\eta(\sigma^{-1}))$. We show the latter identity as a result of finer identities, starting with the following.

Lemma 4.2. *Let η be a composition of n and $\sigma \in S^\eta$. Then*

$$|N_\sigma^+[\preceq]| = \mathrm{inv}(\mathbf{nexc}(\pi_\eta(\sigma^{-1}))).$$

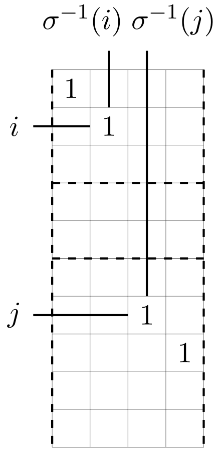

FIGURE 5. A block-column of σ^{-1}.

Proof. Since $\sigma \in S^\eta$, σ is descending on every block-row. Thus σ^{-1} is descending on every block-column, that is if $i < j$ with $\pi_\eta(\sigma^{-1}(i)) = \pi_\eta(\sigma^{-1}(j))$, then $\sigma^{-1}(i) < \sigma^{-1}(j)$; see also Figure 5. But $\sigma^{-1}(i) > \sigma^{-1}(j)$ also implies $\pi_\eta(\sigma^{-1}(i)) \geqslant \pi_\eta(\sigma^{-1}(j))$. Therefore, for $i < j$ we have

$$\sigma^{-1}(i) > \sigma^{-1}(j) \Leftrightarrow \pi_\eta(\sigma^{-1}(i)) > \pi_\eta(\sigma^{-1}(j)). \tag{4.1}$$

By definition, setting $k = \sigma(i)$ and using Eq. (4.1), we get

$$\begin{aligned}|N_\sigma^+[\leq]| &= |\{(k,j) : k < j, \sigma^{-1}(j) < \sigma^{-1}(k), \pi_\eta(\sigma^{-1}(k)) \leqslant \pi_\eta(j), \pi_\eta(\sigma^{-1}(k)) \leqslant \pi_\eta(k)\}| \\ &= |\{(k,j) : k < j, \pi_\eta(\sigma^{-1}(j)) < \pi_\eta(\sigma^{-1}(k)) \leqslant \pi_\eta(j), \pi_\eta(\sigma^{-1}(k)) \leqslant \pi_\eta(k)\}|.\end{aligned} \tag{4.2}$$

Consider the non-exceeding subword of $\pi_\eta(\sigma^{-1})$

$$\mathbf{nexc}(\pi_\eta(\sigma^{-1})) = \pi_\eta(\sigma(i_1)) \cdots \pi_\eta(\sigma(i_m)),$$

where $\pi_\eta(\sigma(i)) \leqslant \pi_\eta(i)$ if and only if $i \in \{i_1, \ldots, i_m\}_< = [n] \backslash \operatorname{Exc}(\pi_\eta(\sigma^{-1}))$. The lemma now follows by comparing Eq. (4.2) with

$$\begin{aligned}\operatorname{inv}(\mathbf{nexc}(\pi_\eta(\sigma^{-1}))) = |\{(i,j) : i < j, \pi_\eta(\sigma^{-1}(j)) < \pi_\eta(\sigma^{-1}(i)), \\ \pi_\eta(\sigma^{-1}(j)) < \pi_\eta(j), \pi_\eta(\sigma^{-1}(i)) < \pi_\eta(i)\}|.\end{aligned}$$

□

We now give a few more definitions that are needed for the next lemma. For $l \in \{2, \ldots, r\}$, following [9, Section 1] we set

$$U_\sigma(l) := \{(i, \sigma(i)) : l \leqslant \pi_\eta(i), \pi_\eta(\sigma(i)) < l\}.$$

Let us further define

$$U_\sigma^{-1}(l) := \{(i, \sigma(i)) : \pi_\eta(i) < l, l \leqslant \pi_\eta(\sigma(i))\}.$$

The statistics $U_\sigma(l)$ and $U_\sigma^{-1}(l)$ count, respectively, the number of ones in certain north-east and south-west quadrants of the grid, see Figure 6 for an example.

For $(j_0, \sigma(j_0)) \in [>]$, we set

$$N_\sigma^-(j_0) := [>] \cap \{(j_0, i) : \sigma(j_0) < i, j_0 < \sigma^{-1}(i)\}, \tag{4.3}$$

and

$$N_\sigma^+[>](j_0) := \{(j_0, i) : \sigma(j_0) < i, \sigma^{-1}(i) < j_0, \pi_\eta(j_0) \leqslant \pi_\eta(i), \pi_\eta(j_0) > \pi_\eta(\sigma(j_0))\}. \tag{4.4}$$

Informally, $N_\sigma^-(j_0)$ (resp. $N_\sigma^+[>](j_0)$) counts the elements of N_σ^- (resp. $N_\sigma^+[>]$) in the j_0-th row of the matrix associated with σ.

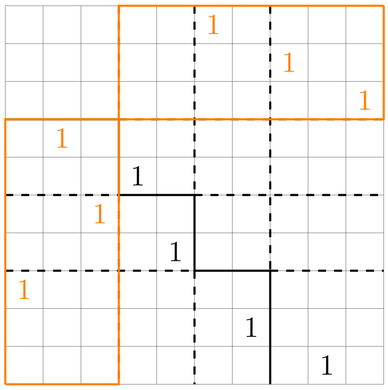

FIGURE 6. $U_\sigma(2)$ (entries equal to 1 in the left orange rectangle) and $U_\sigma^{-1}(2)$ (entries equal to 1 in the right orange rectangle) for $\eta = (3, 2, 2, 3)$ and $\sigma = 68102435179$

Lemma 4.3. *Let η be a composition of n and $\sigma \in S^\eta$. Then*

$$|N_\sigma^+[>]| = \text{imv}(\textbf{exc}(\pi_\eta(\sigma^{-1}))) + |N_\sigma^-| + \text{iexc}(\sigma)$$

Proof. For a fixed excedance $l_0 \in \text{Exc}(\pi_\eta(\sigma^{-1}))$, write $j_0 := \sigma^{-1}(l_0)$ and set

$$M_\sigma^=(j_0) := \{(j_0, \sigma(i)) : \sigma(i) < \sigma(j_0), i < j_0, \pi_\eta(j_0) = \pi_\eta(i), \pi_\eta(\sigma(i)) < \pi_\eta(i)\}$$

and

$$M_\sigma^>(j_0) := \{(j_0, \sigma(i)) : \sigma(i) < \sigma(j_0), i < j_0, \pi_\eta(j_0) < \pi_\eta(i), \pi_\eta(\sigma(i)) < \pi_\eta(i)\}.$$

We prove the lemma in four steps.

1. $\text{imv}(\textbf{exc}(\pi_\eta(\sigma^{-1}))) = \sum_{(j_0,\sigma(j_0))\in[>]} (|M_\sigma^=(j_0)| + |M_\sigma^>(j_0)|)$.
2. $|M_\sigma^=(j_0)| + |M_\sigma^>(j_0)| + |N_\sigma^-(j_0)| + 1 = |U_\sigma(\pi_\eta(j_0))|$.
3. $|U_\sigma(\pi_\eta(j_0))| = |U_\sigma^{-1}(\pi_\eta(j_0))|$.
4. $|U_\sigma^{-1}(\pi_\eta(j_0))| = |N_\sigma^+[>](j_0)|$.

To prove 1 and 2 we will use the following facts.

(i) For $i < j$ we have $\pi_\eta(i) \leqslant \pi_\eta(j)$.
(ii) σ is descending on every block-row, i.e. if $i \neq j$ with $\sigma(j) < \sigma(i)$ and $\pi_\eta(i) = \pi_\eta(j)$ then $j < i$.

Proof of 1. The idea here is to write the number of weak inversions of the exceeding word of $\pi_\eta(\sigma^{-1})$ as a sum of equal pairs and strict inversions. These are, in turn, refined according to the second element of the pair. Indeed, given l_0 and j_0 as before, using (ii) and setting $l := \sigma(i)$ in the definition of $M_\sigma^=(j_0)$, we get

$$\begin{aligned} |M_\sigma^=(j_0)| &= |\{(j_0, \sigma(i)) : \sigma(i) < \sigma(j_0), i < j_0, \pi_\eta(j_0) = \pi_\eta(i), \pi_\eta(\sigma(i)) < \pi_\eta(i)\}| \\ &= |\{(\sigma(i), j_0) : \sigma(i) < \sigma(j_0), \pi_\eta(j_0) = \pi_\eta(i), \pi_\eta(\sigma(i)) < \pi_\eta(i)\}| \\ &= |\{(l, l_0) : l < l_0, \pi_\eta(\sigma^{-1}(l_0)) = \pi_\eta(\sigma^{-1}(l)), \pi_\eta(l) < \pi_\eta(\sigma^{-1}(l))\}|. \end{aligned} \tag{4.5}$$

Similarly,

$$\begin{aligned} |M_\sigma^>(j_0)| &= |\{(j_0, \sigma(i)) : \sigma(i) < \sigma(j_0), i < j_0, \pi_\eta(j_0) < \pi_\eta(i), \pi_\eta(\sigma(i)) < \pi_\eta(i)\}| \\ &= |\{(\sigma(i), j_0) : \sigma(i) < \sigma(j_0), \pi_\eta(j_0) < \pi_\eta(i), \pi_\eta(\sigma(i)) < \pi_\eta(i)\}| \\ &= |\{(l, l_0) : l < l_0, \pi_\eta(\sigma^{-1}(l_0)) < \pi_\eta(\sigma^{-1}(l)), \pi_\eta(l) < \pi_\eta(\sigma^{-1}(l))\}|. \end{aligned} \tag{4.6}$$

The claim follows, as

$$\text{imv}(\textbf{exc}(\pi_\eta(\sigma^{-1}))) = \sum_{l_0} |\{(l, l_0) : l < l_0, \pi_\eta(\sigma^{-1}(l_0)) \leqslant \pi_\eta(\sigma^{-1}(l)), \pi_\eta(l) < \pi_\eta(\sigma^{-1}(l))\}|,$$

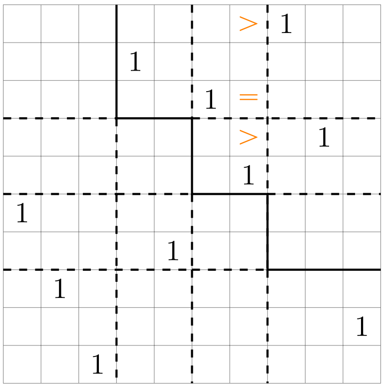

FIGURE 7. For $\eta = (3, 2, 2, 3)$ and $\sigma^{-1} = 84697152103$, pick $l_0 = 5$, so $j_0 = \sigma^{-1}(l_0) = 7$. The cells corresponding to the elements of the set in (4.5) are marked with orange symbols "=" and those corresponding to the elements of the set in (4.6) are marked by orange symbols ">".

where the sum ranges over $l_0 \in \mathrm{Exc}(\pi_\eta(\sigma^{-1}))$.

Proof of 2. We partition $U_\sigma(\pi_\eta(j_0)) = \{(i, \sigma(i)) : \pi_\eta(j_0) \leqslant \pi_\eta(i), \pi_\eta(\sigma(i)) < \pi_\eta(j_0)\}$ as follows:

$$\begin{aligned}
&\underbrace{\{(i,\sigma(i)) : \sigma(i) < \sigma(j_0), i < j_0\} \cap U_\sigma(\pi_\eta(j_0))}_{=:U^1_\sigma(\pi_\eta(j_0))} \\
\cup\ &\underbrace{\{(i,\sigma(i)) : \sigma(i) < \sigma(j_0), i > j_0\} \cap U_\sigma(\pi_\eta(j_0))}_{=:U^2_\sigma(\pi_\eta(j_0))} \\
\cup\ &\underbrace{\{(i,\sigma(i)) : \sigma(i) > \sigma(j_0), i > j_0\} \cap U_\sigma(\pi_\eta(j_0))}_{=:U^3_\sigma(\pi_\eta(j_0))} \\
\cup\ &\{(j_0, \sigma(j_0))\},
\end{aligned}$$

see Figure 8 for an example. Our goal is to rewrite the cardinalities of each of the $U^i_\sigma(\pi_\eta(j_0))$.

$$\begin{aligned}
|U^1_\sigma(\pi_\eta(j_0))| &= |\{(i,\sigma(i)) : \sigma(i) < \sigma(j_0), i < j_0, \pi_\eta(j_0) \leqslant \pi_\eta(i),\ \pi_\eta(\sigma(i)) < \pi_\eta(j_0), \pi_\eta(\sigma(j_0)) < \pi_\eta(j_0)\}| \\
&\overset{(i)}{=} |\{(i,\sigma(i)) : \sigma(i) < \sigma(j_0), i < j_0, \pi_\eta(j_0) = \pi_\eta(i),\ \pi_\eta(\sigma(i)) < \pi_\eta(j_0), \pi_\eta(\sigma(j_0)) < \pi_\eta(j_0)\}| \\
&= |\{(j_0,\sigma(i)) : \sigma(i) < \sigma(j_0), i < j_0, \pi_\eta(j_0) = \pi_\eta(i),\ \pi_\eta(\sigma(i)) < \pi_\eta(i), \pi_\eta(\sigma(j_0)) < \pi_\eta(j_0)\}| \\
&= |M^=_\sigma(j_0)|,
\end{aligned}$$

Similarly,

$$\begin{aligned}
|U^2_\sigma(\pi_\eta(j_0))| =& |\{(i,\sigma(i)) : \sigma(i) < \sigma(j_0), j_0 < i, \pi_\eta(j_0) \leqslant \pi_\eta(i),\ \pi_\eta(\sigma(i)) < \pi_\eta(j_0), \pi_\eta(\sigma(j_0)) < \pi_\eta(j_0)\}| \\
\overset{(ii)}{=}& |\{(j_0,\sigma(i)) : \sigma(i) < \sigma(j_0), j_0 < i, \pi_\eta(j_0) < \pi_\eta(i),\ \pi_\eta(\sigma(i)) < \pi_\eta(j_0), \pi_\eta(\sigma(j_0)) < \pi_\eta(j_0)\}| \\
=& |M^>_\sigma(j_0)|.
\end{aligned}$$

Finally,

$$\begin{aligned}
|U^3_\sigma(\pi_\eta(j_0))| =& |\{(i,\sigma(i)) : \sigma(j_0) < \sigma(i), j_0 < i, \pi_\eta(j_0) \leqslant \pi_\eta(i),\ \pi_\eta(\sigma(i)) < \pi_\eta(j_0), \pi_\eta(\sigma(j_0)) < \pi_\eta(j_0)\}| \\
\overset{(i)}{=}& |\{(j_0,\sigma(i)) : \sigma(j_0) < \sigma(i), j_0 < i, \pi_\eta(\sigma(i)) < \pi_\eta(j_0)\}| \\
=& |\{(j_0,h) : \sigma(j_0) < h, j_0 < \sigma^{-1}(h), \pi_\eta(j_0) > \pi_\eta(h)\}| \\
=& |N^-_\sigma(j_0)|,
\end{aligned}$$

where $h := \sigma(i)$.

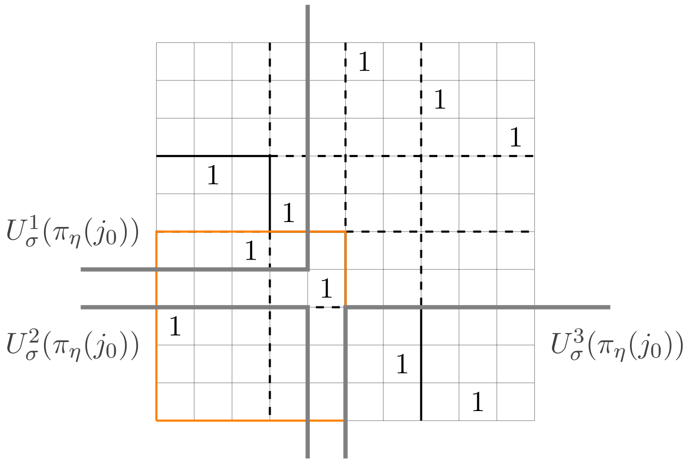

FIGURE 8. Let $\eta = (3, 2, 2, 3)$, $\sigma = 68102435179$ and $j_0 = 7$. $U^i_\sigma(\pi_\eta(j_0))$, $i \in [3]$, is given by the entries equal to 1 in the regions indicated by the grey lines intersected with the orange rectangle

Therefore we obtain

$$\begin{aligned}|U_\sigma(\pi_\eta(j_0))| &= |U^1_\sigma(\pi_\eta(j_0))| + |U^2_\sigma(\pi_\eta(j_0))| + |U^3_\sigma(\pi_\eta(j_0))| + 1\\ &= |M^=_\sigma(j_0)| + |M^>_\sigma(j_0)| + |N^-_\sigma(j_0)| + 1.\end{aligned}$$

Proof of 3. For $n_1, n_2 \in \mathbb{N}$ and a permutation matrix divided into blocks

$$\left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22}\end{array}\right]$$

where each block B_{ij} is an $n_i \times n_j$ matrix, we let $l \in \mathbb{N}_0$ denote the number of entries equal to 1 in B_{21}. That is, $l = |\{(i, \sigma(i)) : \sigma(i) \leqslant n_1 < i\}$ which is also the number of entries equal to 1 in B_{12}, while the number of entries equal to 1 in B_{11} is $n_1 - l$.

Proof of 4. For an excedance $l_0 \in \mathrm{Exc}(\pi_\eta(\sigma^{-1}))$ and $j_0 = \sigma^{-1}(l_0)$, we have

$$|U^{-1}_\sigma(\pi_\eta(j_0))| = |\{(i, \sigma(i)) : \pi_\eta(i) < \pi_\eta(j_0), \pi_\eta(j_0) \leqslant \pi_\eta(\sigma(i)), \pi_\eta(\sigma(j_0)) < \pi_\eta(j_0)\}|.$$

Since $\pi_\eta(\sigma(j_0)) \leqslant \pi_\eta(\sigma(i))$, it follows that $\sigma(j_0) < \sigma(i)$, Thus the above is equal to

$$\begin{aligned}&|\{(j_0, \sigma(i)) : \sigma(j_0) < \sigma(i), i < j_0, \pi_\eta(j_0) \leqslant \pi_\eta(\sigma(i)), \pi_\eta(\sigma(j_0)) < \pi_\eta(j_0)\}|\\ &= |\{(j_0, h) : \sigma(j_0) < h, \sigma^{-1}(h) < j_0, \pi_\eta(j_0) \leqslant \pi_\eta(h), \pi_\eta(\sigma(j_0)) < \pi_\eta(j_0)\}|\\ &= |N^+_\sigma[>](j_0)|,\end{aligned}$$

proving 4.

For $j_0 = \sigma^{-1}(l_0)$, where $l_0 \in \mathrm{Exc}(\pi_\eta(\sigma^{-1})$, it now follows that

$$|N^+_\sigma[>](j_0)| \overset{4.}{=} |U^{-1}_\sigma(\pi_\eta(j_0))| \overset{3.}{=} |U_\sigma(\pi_\eta(j_0))| \overset{2.}{=} |M^=_\sigma(j_0)| + |M^>_\sigma(j_0)| + |N^-_\sigma(j_0)| + 1.$$

Therefore, by Eq. (4.4),

$$\begin{aligned}|N^+_\sigma[>]| &= \sum_{(j,\sigma(j))\in[>]} |N^+_\sigma[>](j)| = \sum_{(j,\sigma(j))\in[>]} |M^=_\sigma(j)| + |M^>_\sigma(j)| + |N^-_\sigma(j)| + 1\\ &= \mathrm{imv}(\mathbf{exc}(\pi_\eta(\sigma^{-1}))) + |N^-_\sigma| + \mathrm{iexc}(\sigma),\end{aligned}$$

where the latter equality follows from 1, Eq. (4.3) and the definition of iexc. □

Proof of Theorem 4.1. Combining Lemma 4.2 and Lemma 4.3, for $\sigma \in S^\eta$ we get

$$\mathrm{den}(\pi_\eta(\sigma^{-1})) = \mathrm{iden}(\sigma).$$

The theorem follows, as by definition $\operatorname{iexc}(\sigma) = \operatorname{exc}(\pi_\eta(\sigma^{-1}))$ and the map defined in Eq. (3.1) is a bijection between S^η and S_η. □

Let $\eta = (\eta_1, \dots, \eta_r)$ be a composition of n. Theorems 2.3 and 4.1 imply that the genus zeta function of the local hereditary order $\Theta = \Theta^\eta$ can be rewritten in terms of the pair of statistics $(\operatorname{maj}, \operatorname{des})$.

Corollary 4.4.

$$Z_{\Theta^\eta}(s) = \frac{\sum_{w \in S_\eta} q^{\operatorname{maj}(w) - ns \operatorname{des}(w)}}{\prod_{i=0}^{n-1}(1 - q^{i-ns})}.$$

The next corollary follows directly from [8, Proposition 2.12] (see also [8, Theorem 1.3]) and establishes a reciprocity property for the genus zeta function of local hereditary orders whose associated composition is a *rectangle* (i.e. all its parts are equal).

Corollary 4.5. *Let $r, m \in \mathbb{N}$ and $\eta = (\underbrace{m, \dots, m}_{r}) =: (m^r)$. Then*

$$Z_{\Theta^\eta}(s)\big|_{q \to q^{-1}} = (-1)^{rm} q^{\frac{rm(m-1)}{2} - mns} Z_{\Theta^\eta}(s).$$

If η is not a rectangle, then $Z_{\Theta^\eta}(s)$ does not satisfy a functional equation of the form

$$Z_{\Theta^\eta}(s)\big|_{q \to q^{-1}} = \pm q^{a - bs} Z_{\Theta^\eta}(s).$$

for $a, b \in \mathbb{N}_0$.

It would be interesting to establish a purely algebraic explanation of this result.

5. Signed and even-signed permutations

In this section, we define signed analogues of the Denert statistic and show that they are, together with the number of absolute excedances, equidistributed with the the flag major index and the number of flag descents over the hyperoctahedral groups. For a suitable definition of type D descents and major indices, we define a type D Denert statistic and number of excedances which are equidistributed over the even-signed permutations.

5.1. Euler-Mahonian statistics on B_n.

Let B_n denote the group of signed permutations on n letters, i.e. permutations of the set $[-n, n]$ such that $\sigma(-i) = -\sigma(i)$ for $i \in [0, n]$. For a signed permutation $\sigma \in B_n$, we use the window notation $\sigma = \sigma(1) \dots \sigma(n)$. By slight abuse of notation, we denote by $\operatorname{des}(\sigma)$ and $\operatorname{maj}(\sigma)$ the type A descent and major index statistics of the signed permutation σ, as defined in Section 2.

Well-known statistics on signed permutations (see for example [1]) include the negative statistics

$$\operatorname{neg}(\sigma) = |\{i \in [n] : \sigma(i) < 0\}|,$$
$$\operatorname{ndes}(\sigma) = \operatorname{des}(\sigma) + \operatorname{neg}(\sigma) \quad \text{and} \quad \operatorname{nmaj}(\sigma) = \operatorname{maj}(\sigma) - \sum_{\sigma(i) < 0} \sigma(i)$$

and the flag statistics

$$\operatorname{fdes}(\sigma) = 2\operatorname{des}(\sigma) + \chi(\sigma(1) < 0) \quad \text{and} \quad \operatorname{fmaj}(\sigma) = 2\operatorname{maj}(\sigma) + \operatorname{neg}(\sigma),$$

where

$$\chi(\sigma(1) < 0) = \begin{cases} 1 & \text{if } \sigma(1) < 0 \\ 0 & \text{otherwise.} \end{cases}$$

In [1] the two pairs of statistics $(\operatorname{nmaj}, \operatorname{ndes})$ and $(\operatorname{fmaj}, \operatorname{fdes})$ were shown to be equidistributed.

Theorem 5.1. [1, Corollary 4.5]

$$\sum_{\sigma\in B_n} q^{\mathrm{nmaj}(\sigma)}t^{\mathrm{ndes}(\sigma)} = \sum_{\sigma\in B_n} q^{\mathrm{fmaj}(\sigma)}t^{\mathrm{fdes}(\sigma)}.$$

Denert's statistic has been extended to signed permutations before (see, e.g., [10]). To the best of our knowledge, none of the type B extensions previously considered gives rise, together with a suitable definition of excedances, to an Euler-Mahonian pair in the sense of Theorem 5.1.

Definition 5.2. [17, Definition 4.1] For $\sigma \in B_n$, we define $|\sigma| = |\sigma(1)|\dots|\sigma(n)| \in S_n$. The absolute excedance number is

$$\mathrm{exc}^{\mathrm{abs}}(\sigma) = \mathrm{exc}(|\sigma|) + \mathrm{neg}(\sigma).$$

We define a Denert statistic for signed permutations as follows.

Definition 5.3. Let $\sigma \in B_n$. The negative Denert statistic is

$$\mathrm{nden}(\sigma) = \mathrm{den}(|\sigma|) - \sum_{\sigma(i)<0} \sigma(i).$$

The following theorem shows that the pairs of statistics $(\mathrm{nden}, \mathrm{exc}^{\mathrm{abs}})$ and $(\mathrm{fmaj}, \mathrm{fdes})$ are equidistributed over the hyperoctahedral groups.

Theorem 5.4.

$$\sum_{\sigma\in B_n} q^{\mathrm{nden}(\sigma)}t^{\mathrm{exc}^{\mathrm{abs}}(\sigma)} = \sum_{\sigma\in B_n} q^{\mathrm{fmaj}(\sigma)}t^{\mathrm{fdes}(\sigma)}.$$

Proof. Writing a signed permutation as a product of an element in the symmetric group and a sign vector yields:

$$\begin{aligned}
\sum_{\sigma\in B_n} q^{\mathrm{nden}(\sigma)}t^{\mathrm{exc}^{\mathrm{abs}}(\sigma)} &= \sum_{\sigma\in B_n} q^{\mathrm{den}(|\sigma|)}q^{-\sum_{\sigma(i)<0}\sigma(i)}t^{\mathrm{exc}(|\sigma|)}t^{\mathrm{neg}(\sigma)} \\
&= \left(\sum_{\sigma\in S_n} q^{\mathrm{den}(\sigma)}t^{\mathrm{exc}(\sigma)}\right)\left(\sum_{J\subseteq[n]}\sum_{j\in J} q^j t\right) \\
&= \left(\sum_{\sigma\in S_n} q^{\mathrm{maj}(\sigma)}t^{\mathrm{des}(\sigma)}\right)\left(\sum_{J\subseteq[n]}\sum_{j\in J} q^j t\right) \\
&= \sum_{\sigma\in B_n} q^{\mathrm{nmaj}(\sigma)}t^{\mathrm{ndes}(\sigma)},
\end{aligned}$$

where the penultimate equality follows from Theorem 2.3. The claim now follows by Theorem 5.1. □

5.2. **Euler-Mahonian statistics on D_n.** We define a type D analogue of Denert's statistic which, together with a suitable definition of an excedance statistic, forms an Euler-Mahonian pair. The Coxeter group D_n is the subgroup of B_n of even-signed permutations,

$$D_n = \{\sigma \in B_n : \mathrm{neg}(\sigma) \equiv 0 \mod 2\}.$$

A negative descent set on D_n and corresponding descent number and major index were defined in [3].

Definition 5.5. [3, Section 3.1] Let $\sigma \in D_n$. The type D negative descent set of σ is

$$\mathrm{DNeg}(\sigma) = \{i \in [n] : \sigma(i) < -1\} \quad \text{and} \quad \mathrm{dneg}(\sigma) = |\,\mathrm{DNeg}(\sigma)|.$$

The corresponding descent and major index statistics are

$$\begin{aligned}\mathrm{ddes}(\sigma) &= \mathrm{des}(\sigma) + \mathrm{dneg}(\sigma),\\ \mathrm{dmaj}(\sigma) &= \mathrm{maj}(\sigma) - \sum_{i\in\mathrm{DNeg}(\sigma)} \sigma(i) - \mathrm{dneg}(\sigma).\end{aligned}$$

Definition 5.6. For $\sigma \in D_n$, we define the number of type D excedances to be

$$\mathrm{dexc}(\sigma) := \mathrm{exc}(|\sigma|) + \mathrm{dneg}(\sigma).$$

Note that the number of type D excedances of $\sigma \in D_n$ differs from the number of absolute excedances of σ if $\sigma(i) = -1$ for some $i \in [n]$.

Definition 5.7. We define Denert's statistic for even-signed permutations as

$$\mathrm{dden}(\sigma) := \mathrm{den}(|\sigma|) - \sum_{i\in\mathrm{DNeg}(\sigma)} \sigma(i) - \mathrm{dneg}(\sigma) = \mathrm{den}(|\sigma|) + \mathrm{nsp}(\sigma),$$

where $\mathrm{nsp}(\sigma) := |\{(i,j) \in [n]\times[n] : i < j, \sigma(i)+\sigma(j) < 0\}|$ denotes the negative sum pairs.

The next theorem shows that $(\mathrm{dden}, \mathrm{dexc})$ and $(\mathrm{dmaj}, \mathrm{ddes})$ are equidistributed over the even-signed permutations.

Theorem 5.8.

$$\sum_{\sigma\in D_n} q^{\mathrm{dden}(\sigma)} t^{\mathrm{dexc}(\sigma)} = \sum_{\sigma\in D_n} q^{\mathrm{dmaj}(\sigma)} t^{\mathrm{ddes}(\sigma)}.$$

Proof. Write D_n as

$$D_n = \bigcup_{\pi\in S_n} \{\tau\pi : \tau \in T\},$$

where $T = \{\tau \in D_n : \mathrm{des}(\tau) = 0\}$ and the union is disjoint. Then

$$\sum_{\sigma\in D_n} q^{\mathrm{dden}(\sigma)} t^{\mathrm{dexc}(\sigma)} = \sum_{\pi\in S_n}\sum_{\tau\in T} q^{\mathrm{den}(|\tau\pi|) - \sum_{i\in\mathrm{DNeg}(\tau\pi)} \tau\pi(i) - \mathrm{dneg}(\tau\pi)} t^{\mathrm{exc}(|\tau\pi|)+\mathrm{dneg}(\tau\pi)}. \quad (5.1)$$

It is easy to see that $\sum_{i\in\mathrm{DNeg}(\tau\pi)} \tau\pi(i) = \sum_{i\in\mathrm{DNeg}(\tau)} \tau(i)$ and $\mathrm{dneg}(\tau\pi) = \mathrm{dneg}(\tau)$ for any $\pi \in S_n$ and $\tau \in T$. Thus Eq. (5.1) is equal to

$$\sum_{\tau\in T} q^{-\sum_{i\in\mathrm{DNeg}(\tau)} \tau(i) - \mathrm{dneg}(\tau)} t^{\mathrm{dneg}(\tau)} \sum_{\pi\in S_n} q^{\mathrm{den}(\pi)} t^{\mathrm{exc}(\pi)}.$$

By Theorem 2.3, this is equal to

$$\begin{aligned}&= \sum_{\tau\in T} q^{-\sum_{i\in\mathrm{DNeg}(\tau)} \tau(i) - \mathrm{dneg}(\tau)} t^{\mathrm{dneg}(\tau)} \sum_{\pi\in S_n} q^{\mathrm{maj}(\pi)} t^{\mathrm{des}(\pi)}\\ &= \sum_{\sigma\in D_n} q^{\mathrm{dmaj}(\sigma)} t^{\mathrm{ddes}(\sigma)},\end{aligned}$$

which proves the theorem. □

6. Final remarks

6.1. **Hadamard products.** By a formula due to MacMahon [16, §462, Vol. 2, Ch. IV, Sect. IX] and Theorem 4.1, it turns out that genus zeta functions as in Corollary 4.4, viewed as rational functions in q and q^{-ns} are closely related to Hadamard products of the rational functions expressing genus zeta functions of maximal orders (i.e. orders whose local type is a composition with one part). In the following, given rational functions $F(y)$ and $G(y)$, we denote with $(F \star G)(y)$ their Hadamard product. Then

$$W_\eta(x,y) = (1-x^n y) \mathop{\bigstar}_{i=1}^{r} W_{(\eta_i+1)}(x,y) = (1-x^n y) \mathop{\bigstar}_{i=1}^{r} \left(\prod_{0\leqslant j\leqslant \eta_i} \frac{1}{1-x^j y} \right), \quad (6.1)$$

where the Hadamard product is taken with respect to y.

At present, we are not aware of an algebraic interpretation, say in terms of factorisation of ideals in Θ^η, of the Hadamard product in Eq. (6.1).

Certain orbit Dirichlet series exhibit a similar behaviour; cf. [8, Proposition 1.2]. An algebraic framework for interpreting Hadamard products of closely related generating functions was recently developed by Gessel and Zhuang [13].

6.2. **Factorisation.** It is well known that classical Eulerian polynomials over S_n have all real, simple negative roots and that -1 is a root if and only n is even; see [12]. It was proved in [8, Lemma 2.7] that this generalises to a factorisation of the q-Carlitz polynomial for n even. In the same paper, it also was conjectured that a similar factorisation result should hold for the polynomials giving the joint distribution of $(\mathrm{des}, \mathrm{maj})$ over multiset permutations associated with compositions which are rectangles and satisfy certain conditions (see [8, Conjecture B]). Han's result Theorem 2.3 and Theorem 4.1 allow for reformulations of this conjecture in terms of the pair of statistics $(\mathrm{exc}, \mathrm{den})$ over multiset permutations and in terms of Denert's original statistic over η-admissible permutations. More precisely, the conjecture revolves around the existence of so-called unitary factors of Euler-Mahonian polynomials. A nonconstant polynomial $f \in \mathbb{Z}[x, y]$ is called *unitary* if there exists $F \in \mathbb{Z}[Y]$ such that $f(x, y) = F(x^a y^b)$ for some $a, b \in \mathbb{N}_0$ and all complex roots of F have absolute value 1.

Conjecture A. *Let η be a composition. Then the polynomial of the joint distribution of $(\mathrm{den}, \mathrm{exc})$ over S_η has a unitary factor if and only if $\eta = (m^r)$ is a rectangle, with r even and m odd. In this case,*

$$\sum_{w \in S_\eta} x^{\mathrm{den}(w)} y^{\mathrm{exc}(w)} = \left(1 + x^{\frac{rm}{2}} y\right) f_0^\eta(x, y)$$

where $f_0^\eta(x, y)$ has no unitary factor.

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