Séminaire Lotharingien de Combinatoire **XX** (2024)
Article #YY, 12 pp.

Proceedings of the 36th Conference on Formal Power Series and Algebraic Combinatorics (Bochum)

From coloured permutations to Hadamard products and zeta functions

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Abstract. We devise a constructive method for computing explicit combinatorial formulae for Hadamard products of certain rational generating functions. The latter arise naturally when studying so-called ask zeta functions of direct sums of modules of matrices or class- and orbit-counting zeta functions of direct products of groups. Our method relies on shuffle compatibility of coloured permutation statistics and coloured quasisymmetric functions, extending recent work of Gessel and Zhuang.

Keywords: Coloured permutations, permutation statistics, Hadamard products, shuffle compatibility, average sizes of kernels, zeta functions

1 Introduction

Permutation statistics are functions defined on permutations and their generalisations. Studying the behaviour of said functions on sets of permutations is a classical theme in algebraic and enumerative combinatorics. The origins of permutation statistics can be traced back to work of Euler and MacMahon. The past decades saw a flurry of further developments in the area; see e.g. [2, 32] and references therein. Recently, Gessel and Zhuang [17] developed an algebraic framework for systematically studying so-called shuffle-compatible permutation statistics by means of associated shuffle algebras. In their work, quasisymmetric functions and Hadamard products of rational generating functions played key roles.

Numerous types of zeta functions have been employed in the study of enumerative problems surrounding algebraic structures. L. Solomon [31] introduced zeta functions associated with integral representations and, in another influential paper, Grunewald, Segal, and Smith [18] initiated the study of zeta functions associated with (nilpotent and pro-p) groups. Following [18], a variety of methods have been developed and applied to

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predict the behaviour and study symmetries of zeta functions associated with algebraic structures, and to produce explicit formulae. Theoretical work of this type often employs a blend of combinatorics and p-adic integration; see [35] for a survey. On the practical side, a range of effective methods have been devised and used to symbolically compute zeta functions of algebraic structures; see [25] and the references therein.

A common feature of zeta functions $\zeta_G(s)$ attached to algebraic structures G (e.g. groups) in the literature is that they often admit an Euler factorisation $\zeta_G(s) = \prod_p \zeta_{G,p}(s)$ into so-called local factors indexed by primes p. Deep results from p-adic integration often guarantee that these local factors are rational in p^{-s}, i.e. of the form $\zeta_{G,p}(s) = W_p(p^{-s})$ for some $W_p(Y) \in \mathbb{Q}(Y)$. A key theme is then to study how the $W_p(Y)$ vary with the prime p. In a surprising number of cases of interest, deep *uniformity results* ensure the existence of a *single* bivariate rational function $W(X,Y)$ such that $\zeta_{G,p}(s) = W(p, p^{-s})$ for all primes p (perhaps excluding a finite number of exceptions). In such situations, understanding our zeta function is tantamount to understanding $W(X,Y)$.

In this context, permutation statistics (and, more generally, combinatorial objects) have recently found spectacular applications, in particular when it comes to describing the numerators of the rational functions $W(X,Y)$ from above; see, for instance, [1, 12, 9, 8, 10, 34]. Conversely, the need for combinatorial descriptions of such zeta functions gave rise to new directions in the study of permutation statistics and, more generally, combinatorial objects; see, e.g. [4, 5, 11, 13, 16, 14, 33].

In the spirit of this line of research, in the present work we relate permutation statistics and *ask zeta functions*. Introduced in [26] and developed further in [27, 30, 7], ask zeta functions are generating functions encoding average sizes of kernels in suitable modules of matrices. One motivation for studying these functions comes from group theory. Indeed, for groups with a sufficiently powerful Lie theory, the enumeration of linear orbits and conjugacy classes boils down to determining average sizes of kernels within matrix Lie algebras—this is essentially the orbit-counting lemma.

Amidst a plethora of algebraically-defined zeta functions, ask zeta functions stand out as particularly amenable to combinatorial methods. Indeed, natural operations at the level of the modules (or groups) often translate into natural operations of corresponding rational generating functions. In particular, ask zeta functions of direct sums of modules are Hadamard products of the ask zeta function of the summands.

In this extended abstract (and forthcoming preprint [6]) we answer some of the questions from [30] and give a constructive algorithm (based on *coloured shuffle compatibility* of permutation statistics) to compute Hadamard products of certain ask zeta functions. Our results have corollaries pertaining to generating functions enumerating orbits of finite direct products of groups within various infinite families.

2 Hadamard products and coloured configurations

In this section, we provide a self-contained account of our main result pertaining to Hadamard products of suitable rational generating functions. Its proof relies on the coloured shuffle compatibility of certain permutation statistics and the structure of associated coloured shuffle algebras. We will describe the latter in Section 3.

Coloured permutations and descents. We consider coloured permutations with symbols taken from the poset $\Sigma = \{1 < 2 < \dots\}$ and colours taken from $\Gamma = \{0 > 1 > 2 > \cdots\}$. Let $\boldsymbol{a} = \sigma^\gamma = \sigma_1^{\gamma_1} \cdots \sigma_n^{\gamma_n}$ be a coloured permutation. We write $|\boldsymbol{a}| = n$ for the length of $\boldsymbol{a}$. We further write $\operatorname{sym}(\boldsymbol{a}) = \{\sigma_1, \dots, \sigma_n\}$, $\operatorname{pal}(\boldsymbol{a}) = \{\gamma_1, \dots, \gamma_n\}$, and $\operatorname{pal}^*(\boldsymbol{a}) = \operatorname{pal}(\boldsymbol{a}) \setminus \{0\}$. On the set of Γ-coloured positive integers, consider the total order

$$\cdots < 1^1 < 2^1 < \cdots < 1^0 < 2^0 < \dots;$$

that is, $\sigma_1^{\gamma_1} < \sigma_2^{\gamma_2}$ if and only if $\gamma_1 = \gamma_2$ and $\sigma_1 < \sigma_2$, or if $\gamma_1 > \gamma_2$ in $\mathbb{Z}$ (equivalently: $\gamma_1 < \gamma_2$ in Γ). This is the usual *colour order*, corresponding to the left lexicographic order on $\Gamma \times \Sigma$. The *descent set* of $\boldsymbol{a}$ as above consists of all $i \in [n-1]$ such that $\sigma_i^{\gamma_i} > \sigma_{i+1}^{\gamma_{i+1}}$ together with 0 whenever $\gamma_1 \neq 0$. The *descent number* and *comajor index* are defined as always as functions of the descent set: $\operatorname{des}(\boldsymbol{a}) = |\operatorname{Des}(\boldsymbol{a})|$ and $\operatorname{comaj}(\boldsymbol{a}) = \sum_{i \in \operatorname{Des}(\boldsymbol{a})}(n-i)$.

Coloured configurations. Let $\mathcal{A}$ be the set of all coloured permutations, and let $\mathbb{Z}\mathcal{A}$ be the free abelian group with basis $\mathcal{A}$. We call elements of $\mathbb{Z}\mathcal{A}$ *coloured configurations*. These elements are of the form $f = \sum_{\boldsymbol{a} \in \mathcal{A}} f_{\boldsymbol{a}} \boldsymbol{a}$, where each $f_{\boldsymbol{a}}$ belongs to $\mathbb{Z}$ and almost all $f_{\boldsymbol{a}}$ are zero. Write $\operatorname{supp}(f) = \{\boldsymbol{a} \in \mathcal{A} : f_{\boldsymbol{a}} \neq 0\}$, $\operatorname{sym}(f) = \bigcup_{\boldsymbol{a} \in \operatorname{supp}(f)} \operatorname{sym}(\boldsymbol{a})$, and $\operatorname{pal}^*(f) = \bigcup_{\boldsymbol{a} \in \operatorname{supp}(f)} \operatorname{pal}^*(\boldsymbol{a})$. We call $f, g \in \mathbb{Z}\mathcal{A}$ *strongly disjoint* if $\operatorname{sym}(f) \cap \operatorname{sym}(g) = \varnothing = \operatorname{pal}^*(f) \cap \operatorname{pal}^*(g)$. For $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{A}$, let $\boldsymbol{a} \sqcup\!\sqcup \boldsymbol{b} \in \mathbb{Z}\mathcal{A}$ be the sum over all shuffles of $\boldsymbol{a}$ and $\boldsymbol{b}$. We extend $\sqcup\!\sqcup$ to a bi-additive product on $\mathbb{Z}\mathcal{A}$.

Labels. Let $U = \{\pm X^k : k \in \mathbb{Z}\}$, viewed as a subgroup of the multiplicative group of the field $\mathbb{Q}(X)$. For $\alpha \colon \Gamma \to U$, write $\operatorname{supp}(\alpha) = \{c \in \Gamma : \alpha(c) \neq 1\}$ and, for $\boldsymbol{a} = \sigma_1^{\gamma_1} \cdots \sigma_n^{\gamma_n}$ as above, let $\alpha(\boldsymbol{a}) = \prod_{i=1}^n \alpha(\gamma_i)$. A *labelled coloured configuration* is a pair (f, α), where $f \in \mathbb{Z}\mathcal{A}$ and $\alpha \colon \Gamma \to U$ satisfies $\operatorname{supp}(\alpha) \subseteq \operatorname{pal}^*(f)$. Given labelled coloured configurations (f, α) and (g, β) such that f and g are strongly disjoint, the pair $(f \sqcup\!\sqcup g, \alpha\beta)$ is a labelled coloured configuration too. (Here, $\alpha\beta$ denotes the pointwise product of α and β.)

Equivalence. Let (f, α) be a labelled coloured configuration. Let $\phi \colon \operatorname{sym}(f) \to S$ and $\psi \colon \operatorname{pal}^*(f) \to P$ be order-preserving bijections onto finite subsets of Σ and $\Gamma \setminus \{0\}$, respectively. Given ϕ and ψ, define a labelled coloured permutation (f', α') as follows. For $\boldsymbol{a} \in \operatorname{supp}(f)$, say $\boldsymbol{a} = \sigma_1^{\gamma_1} \cdots \sigma_n^{\gamma_n}$, write $\boldsymbol{a}' = \phi(\sigma_1)^{\psi(\gamma_1)} \cdots \phi(\sigma_n)^{\psi(\gamma_n)}$. Define $f' = \sum_{\boldsymbol{a} \in \operatorname{supp}(f)} f_{\boldsymbol{a}} \boldsymbol{a}'$. The support of α' is P and $\alpha'(\psi(c)) = \alpha(c)$ for $c \in \operatorname{pal}^*(f)$. We call (f, α) and each (f', α') (as ϕ and ψ range over possible choices) *equivalent*, written $(f, \alpha) \asymp (f', \alpha')$. This defines an equivalence relation on labelled coloured configurations.

Rational functions. Given a labelled coloured configuration (f,α) and $\varepsilon \in \mathbb{Z}$, we define a rational formal power series

$$W^{\varepsilon}_{f,\alpha} = W^{\varepsilon}_{f,\alpha}(X,Y) = \sum_{\boldsymbol{a}\in\mathrm{supp}(f)} f_{\boldsymbol{a}} \frac{\alpha(\boldsymbol{a}) X^{\varepsilon\,\mathrm{comaj}(\boldsymbol{a})} Y^{\mathrm{des}(\boldsymbol{a})}}{(1-Y)(1-X^{\varepsilon}Y)\cdots(1-X^{\varepsilon|\boldsymbol{a}|}Y)} \in \mathbb{Q}(X)[\![Y]\!].$$

Note that, by construction, if $(f,\alpha) \asymp (f',\alpha')$, then $W^{\varepsilon}_{f,\alpha} = W^{\varepsilon}_{f',\alpha'}$ for all $\varepsilon \in \mathbb{Z}$.

Example 2.1. Let $f = 1^0 + 1^1$. Let $\alpha\colon \Gamma \to U$ with $\mathrm{supp}(\alpha) \subseteq \mathrm{pal}^*(f) = \{1\}$. Then

$$W^{\varepsilon}_{f,\alpha} = \frac{1+\alpha(1)X^{\varepsilon}Y}{(1-Y)(1-X^{\varepsilon}Y)}.$$

Recall that the *Hadamard product* of two formal power series $A(Y) = \sum_{k=0}^{\infty} a_k Y^k$ and $B(Y) = \sum_{k=0}^{\infty} b_k Y^k$ is the power series $A(Y) *_Y B(Y) = \sum_{k=0}^{\infty} a_k b_k Y^k$.

Theorem 2.2. *Let (f,α) and (g,β) be labelled coloured configurations such that f and g are strongly disjoint. Then $W^{\varepsilon}_{f,\alpha} *_Y W^{\varepsilon}_{g,\beta} = W^{\varepsilon}_{f\sqcup g,\,\alpha\beta}$ for each $\varepsilon \in \mathbb{Z}$.*

Example 2.3. Let $f = 1^0 + 1^1$ and $g = 2^0 + 2^2$. Then

$$\begin{aligned} f \sqcup g &= (1^0 + 1^1) \sqcup (2^0 + 2^2) \\ &= 1^0 \sqcup 2^0 + 1^0 \sqcup 2^2 + 1^1 \sqcup 2^0 + 1^1 \sqcup 2^2 \\ &= 1^0 2^0 + 2^0 1^0 + 1^0 2^2 + 2^2 1^0 + 1^1 2^0 + 2^0 1^1 + 1^1 2^2 + 2^2 1^1. \end{aligned}$$

Let α and β satisfy $\mathrm{supp}(\alpha) \subseteq \{1\}$ and $\mathrm{supp}(\beta) \subseteq \{2\}$. By Theorem 2.2,

$$\begin{aligned} W^{\varepsilon}_{f,\alpha} *_Y W^{\varepsilon}_{g,\beta} &= \frac{1+\alpha(1)X^{\varepsilon}Y}{(1-Y)(1-X^{\varepsilon}Y)} *_Y \frac{1+\beta(2)X^{\varepsilon}Y}{(1-Y)(1-X^{\varepsilon}Y)} \\ &= \frac{1 + (1+\alpha(1)+\beta(2))X^{\varepsilon}Y + (\alpha(1)+\beta(2)+\alpha(1)\beta(2))X^{2\varepsilon}Y + \alpha(1)\beta(2)X^{3\varepsilon}Y^2}{(1-Y)(1-X^{\varepsilon}Y)(1-X^{2\varepsilon}Y)} \\ &= W^{\varepsilon}_{f\sqcup g,\,\alpha\beta}. \end{aligned}$$

Theorem 2.2 implies, in particular, that for each <u>fixed</u> $\varepsilon \in \mathbb{Z}$, the set

$$\left\{ W^{\varepsilon}_{f,\alpha} : (f,\alpha) \text{ is a coloured configuration} \right\}$$

is closed under Hadamard products in Y. Indeed, given coloured configurations (f,α) and (g,β), we can find (g',β') such that f and g' are strongly disjoint and $(g,\beta) \asymp (g',\beta')$. In that case, $W^{\varepsilon}_{f,\alpha} *_Y W^{\varepsilon}_{g,\beta} = W^{\varepsilon}_{f,\alpha} *_Y W^{\varepsilon}_{g',\beta'} = W^{\varepsilon}_{f\sqcup g',\,\alpha\beta'}$ is computed by the preceding theorem.

In Section 4, we will apply Theorem 2.2 to provide explicit combinatorial descriptions of Hadamard products of ask, class- and orbit-counting zeta functions.

3 Coloured shuffle compatibility

For technical reasons, in this section, we will only consider coloured permutations with colours drawn from $\{0 > 1 > \cdots > r-1\}$ (for sufficiently large r). For clarity, we occasionally refer to these as *r-coloured permutations*. A *coloured permutation statistic* is a function st defined on the set of coloured permutations such that given a coloured permutation σ^γ, if π is a permutation of the same length as σ and with the same relative order, then $\mathrm{st}(\sigma^\gamma) = \mathrm{st}(\pi^\gamma)$. Given coloured permutation statistics $\mathrm{st}_1, \ldots, \mathrm{st}_k$, we regard the tuple $(\mathrm{st}_1, \ldots, \mathrm{st}_k)$ as a coloured permutation statistic via $(\mathrm{st}_1, \ldots, \mathrm{st}_k)(\boldsymbol{a}) = (\mathrm{st}_1(\boldsymbol{a}), \ldots, \mathrm{st}_k(\boldsymbol{a}))$. Given a coloured permutation $\boldsymbol{a} = \sigma^\gamma = \sigma_1^{\gamma_1} \cdots \sigma_n^{\gamma_n}$, let $\mathrm{col}_j(\boldsymbol{a}) := |\{i \in [n] : \gamma_i = j\}|$. The *colour vector* of a $\boldsymbol{a}$ is $\mathbf{col}(\boldsymbol{a}) = (\mathrm{col}_0(\boldsymbol{a}), \ldots, \mathrm{col}_{r-1}(\boldsymbol{a}))$; this is a weak composition of n. The functions $\mathbf{col}$, des, and comaj are coloured permutation statistics.

Recall from [17, 22] that a (coloured) permutation statistic st is *shuffle compatible* if for coloured permutations $\boldsymbol{a}$ and $\boldsymbol{b}$ on disjoint sets of symbols, the multiset $\{\{\mathrm{st}(\boldsymbol{c}) : \boldsymbol{c} \in \boldsymbol{a} \sqcup\!\sqcup \boldsymbol{b}\}\}$ only depends on $\mathrm{st}(\boldsymbol{a}), \mathrm{st}(\boldsymbol{b})$ and the lengths of $\boldsymbol{a}$ and $\boldsymbol{b}$. (Here, $\boldsymbol{a} \sqcup\!\sqcup \boldsymbol{b}$ denotes the set of all coloured permutations obtained as shuffles of $\boldsymbol{a}$ and $\boldsymbol{b}$.) Generalising [17, 22], we associate a shuffle algebra $\mathcal{A}_{\mathrm{st}}^{(r)}$ over $\mathbb{Q}$ to a shuffle-compatible coloured permutation statistic st as follows. First, st defines an equivalence relation $\sim_{\mathrm{st}}$ on r-coloured permutations via $\boldsymbol{a} \sim_{\mathrm{st}} \boldsymbol{b}$ if and only if $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same length and $\mathrm{st}(\boldsymbol{a}) = \mathrm{st}(\boldsymbol{b})$; we refer to this as st-*equivalence*. We write $[\boldsymbol{a}]_{\mathrm{st}}$ to denote the st-equivalence class of $\boldsymbol{a}$. As a $\mathbb{Q}$-vector space $\mathcal{A}_{\mathrm{st}}^{(r)}$ has a basis given by the st-equivalence classes of r-coloured permutations. The multiplication is given by linearly extending the rule

$$[\boldsymbol{a}]_{\mathrm{st}}\,[\boldsymbol{b}]_{\mathrm{st}} = \sum_{\boldsymbol{c} \in \boldsymbol{a} \sqcup\!\sqcup \boldsymbol{b}} [\boldsymbol{c}]_{\mathrm{st}},$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are r-coloured permutations on disjoint sets of symbols. (Thanks to the shuffle compatibility of st, this yields a well-defined multiplication on $\mathcal{A}_{\mathrm{st}}^{(r)}$.)

The main shuffle algebra of interest to us is the one attached to $(\mathrm{des}, \mathrm{comaj}, \mathbf{col})$. Let $p_0, \ldots, p_{r-1}, x, t$ be commuting variables over $\mathbb{Q}$; write $\boldsymbol{p} = (p_0, \ldots, p_{r-1})$. For a ring R, let $R[\![t*]\!]$ denote the ring $R[\![t]\!]$ with multiplication given by the Hadamard product in t.

Theorem 3.1.

(a) The tuple of statistics $(\mathrm{des}, \mathrm{comaj}, \mathbf{col})$ *is shuffle compatible.*

(b) The linear map on $\mathcal{A}^{(r)}_{(\mathrm{des},\mathrm{comaj},\mathbf{col})}$ *defined by*

$$[\boldsymbol{a}]_{(\mathrm{des},\mathrm{comaj},\mathbf{col})} \mapsto \begin{cases} \frac{\boldsymbol{p}^{\mathbf{col}(\boldsymbol{a})} x^{\mathrm{comaj}(\boldsymbol{a})} t^{\mathrm{des}(\boldsymbol{a})+1}}{(1-t)(1-xt)\cdots(1-x^{|\boldsymbol{a}|}t)} z^{|\boldsymbol{a}|}, & \text{if } |\boldsymbol{a}| \geqslant 1 \\ \frac{1}{1-t}, & \text{if } |\boldsymbol{a}| = 0 \end{cases}$$

is an isomorphism from the (r-coloured) shuffle algebra of (des, comaj, **col**) *onto the subalgebra of* $\mathbb{Q}[p_0, p_1, \ldots, p_{r-1}, z, x][\![t*]\!]$ *spanned by*

$$\left\{\frac{1}{1-t}\right\} \bigcup \left\{\frac{p_0^{c_0} \cdots p_{r-1}^{c_{r-1}} x^k t^{j+1}}{(1-t)(1-xt)\cdots(1-x^n t)} z^n\right\}_{n\geqslant 1,\, j\in[0,n],\, c_0,\ldots,c_{r-1}\in[0,r-1],\, k\in\left[\binom{j+1}{2}, nj-\binom{j}{2}\right]}.$$

While we omit the proof of the preceding theorem here, we should like to take this opportunity to provide a brief overview of its key steps and ingredients. The *coloured descent set* of a coloured permutation $\boldsymbol{a} = \sigma_1^{\gamma_1} \cdots \sigma_n^{\gamma_n}$ is defined as

$$\mathrm{sDes}(\boldsymbol{a}) = \Big\{(i, \gamma_i) : i \in [n-1],\ \gamma_i \neq \gamma_{i+1} \text{ or } (\gamma_i = \gamma_{i+1} \text{ and } \sigma_i > \sigma_{i+1})\Big\} \cup \Big\{(n, \gamma_n)\Big\}.$$

This coloured permutation statistic, which was introduced by Mantaci and Reutenauer [21] while studying a coloured generalisation of Solomon's descent algebra, is shuffle compatible. Moreover, it refines the tuple (des, comaj, **col**). As a consequence, the algebra $\mathcal{A}^{(r)}_{(\mathrm{des},\mathrm{comaj},\mathbf{col})}$ is naturally a quotient of $\mathcal{A}^{(r)}_{\mathrm{sDes}}$.

Let $x_i^{(j)}$ for $i = 1, 2, \ldots$ and $j = 0, 1, \ldots, r-1$ be independent (commuting) variables. We write $\boldsymbol{x}^{(j)} = (x_1^{(j)}, x_2^{(j)}, \ldots)$. The *coloured quasisymmetric function* attached to an r-coloured permutation $\boldsymbol{a} = \sigma^{\gamma}$ of length n is

$$F_{\boldsymbol{a}}(\boldsymbol{x}^{(0)}, \ldots, \boldsymbol{x}^{(r-1)}) = \sum_{\substack{1\leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n \\ j\in \mathrm{Des}^*(\boldsymbol{a}) \Rightarrow i_j < i_{j+1}}} x_{i_1}^{(\gamma_1)} x_{i_2}^{(\gamma_2)} \cdots x_{i_n}^{(\gamma_n)},$$

where $\mathrm{Des}^*(\boldsymbol{a}) = \mathrm{Des}(\boldsymbol{a}) \setminus \{0\}$. This is a (homogeneous) formal power series of degree n in the variables $\boldsymbol{x}^{(0)}, \ldots, \boldsymbol{x}^{(r-1)}$. These functions were first introduced in [24]; see also [19, 22, 23]. The space $\mathrm{QSym}^{(r)}$ spanned by all such coloured quasisymmetric functions forms a $\mathbb{Q}$-algebra. It turns out that $\mathrm{QSym}^{(r)}$ and $\mathcal{A}^{(r)}_{\mathrm{sDes}}$ are canonically isomorphic. Our proof of Theorem 3.1 is based on a judicious choice of specialisations of coloured quasisymmetric functions; cf. [22, §4.4]. Theorem 2.2 is a consequence of Theorem 3.1.

4 Applications to zeta functions

4.1 Ask, class-counting, and orbit-counting zeta functions

The main purpose of the present section is to recall four families of zeta functions associated with algebraic structures (Examples 4.1–4.6). These will feature in our applications of Theorem 2.2 in Section 4.2. For further details, see [26, 27]. Rings will be assumed to be commutative and unital. In order to maintain consistency with the literature, we regard $d \times e$ matrices over a ring R as homomorphisms $R^d \to R^e$ acting by right multiplication.

Global ask zeta functions. Given a module $M \subseteq \mathrm{M}_{d\times e}(\mathbb{Z})$ of integral matrices, for each $n \geqslant 1$, let $M_n \subseteq \mathrm{M}_{d\times e}(\mathbb{Z}/n\mathbb{Z})$ denote the reduction of M modulo n. The *(global) ask zeta function* of M is the Dirichlet series $\zeta^{\mathrm{ask}}_M(s) = \sum_{n=1}^{\infty} a_n(M) n^{-s}$, where $a_n(M) \in \mathbb{Q}$ denotes the average size of the kernel of matrices in M_n. By the Chinese remainder theorem, $\zeta^{\mathrm{ask}}_M(s) = \prod_p \zeta^{\mathrm{ask}}_{M_p}(s)$ (Euler product), where the product is taken over all primes p and the local factor at p is given by $\zeta^{\mathrm{ask}}_{M_p}(s) = \sum_{k=0}^{\infty} a_{p^k}(M) p^{-ks}$, a power series in p^{-s}. Drawing upon deep results from p-adic integration and the theory of zeta functions of algebraic structures, it is known that each $\zeta^{\mathrm{ask}}_{M_p}(s)$ is rational in p^{-s}.

Local ask zeta functions. It is often advantageous to bypass global structures altogether and directly study variants of the local factors from above. Let $\mathfrak{O}$ be a compact discrete valuation ring. Let $\mathfrak{P}$ be the maximal ideal of $\mathfrak{O}$ and let q denote the size of the residue field $\mathfrak{O}/\mathfrak{P}$. Such rings $\mathfrak{O}$ are precisely the valuation rings of non-Archimedean local fields. Examples include the p-adic integers $\mathbb{Z}_p$ (in which case $\mathfrak{O}/\mathfrak{P} \cong \mathbb{F}_p$) and the ring $\mathbb{F}_q[\![z]\!]$ of formal power series over $\mathbb{F}_q$ (in which case $\mathfrak{P} = z\mathbb{F}_q[\![z]\!]$).

Given a module of matrices $M \subseteq \mathrm{M}_{d\times e}(\mathfrak{O})$, its associated *(local) ask zeta function* is the formal power series $\mathsf{Z}^{\mathrm{ask}}_M(Y) = \sum_{k=0}^{\infty} \alpha_k(M) Y^k$, where $\alpha_k(M)$ denotes the average size of the kernels within the reduction of M modulo $\mathfrak{P}^k$.

Example 4.1. $\mathsf{Z}^{\mathrm{ask}}_{\mathrm{M}_{d\times e}(\mathfrak{O})}(Y) = \frac{1-q^{-e}Y}{(1-Y)(1-q^{d-e}Y)}$; see [26, Prop. 1.5].

Example 4.2. Let $\mathfrak{O}$ have characteristic distinct from 2. Let $\mathfrak{so}_d(\mathfrak{O})$ be the module of antisymmetric $d \times d$ matrices over $\mathfrak{O}$. By [26, Prop. 5.11], $\mathsf{Z}^{\mathrm{ask}}_{\mathfrak{so}_d(\mathfrak{O})}(Y) = \frac{1-q^{1-d}Y}{(1-Y)(1-qY)}$.

Example 4.3. Let $\mathfrak{n}_d(\mathfrak{O})$ be the module of strictly upper triangular $d \times d$ matrices over $\mathfrak{O}$. By [26, Prop. 5.15(i)], $\mathsf{Z}^{\mathrm{ask}}_{\mathfrak{n}_d(\mathfrak{O})}(Y) = \frac{(1-Y)^{d-1}}{(1-qY)^d}$.

Class- and orbit-counting zeta functions. Let $\mathfrak{O}$ be a compact discrete valuation ring as above. Let G be a linear group scheme over $\mathfrak{O}$, with a given embedding into $d \times d$ matrices. The *orbit-counting zeta function* of G is the generating function $\mathsf{Z}^{\mathrm{oc}}_{\mathsf{G}}(Y) = \sum_{k=0}^{\infty} b_k(\mathsf{G}) Y^k$, where $b_k(\mathsf{G})$ denotes the number of orbits of the (finite) matrix group $\mathsf{G}(\mathfrak{O}/\mathfrak{P}^k)$ on its natural module $(\mathfrak{O}/\mathfrak{P}^k)^d$. The *class-counting zeta function* of G is the generating function $\mathsf{Z}^{\mathrm{cc}}_{\mathsf{G}}(Y) = \sum_{k=0}^{\infty} c_k(\mathsf{G}) Y^k$, where $c_k(\mathsf{G})$ denotes the number of conjugacy classes of $\mathsf{G}(\mathfrak{O}/\mathfrak{P}^k)$. Class-counting zeta functions go back to work of du Sautoy [15]. As shown in [26, 27], subject to restrictions on the residue field size q of $\mathfrak{O}$, class- and orbit-counting zeta functions of G are instances of ask zeta functions associated with modules of matrices over $\mathfrak{O}$. When passing between ask and class-counting zeta functions, one often needs to apply a transformation $Y \leftarrow q^m Y$ for a suitable integer m; see below for an example and cf. Lemma 4.7.

Example 4.4. Suppose that the residue field size q of $\mathfrak{O}$ is odd. By exponentiation, the free class-2-nilpotent Lie algebra on d generators over $\mathfrak{O}$ gives rise to a group scheme $\mathsf{F}_{2,d}$ over $\mathfrak{O}$. We may view $\mathsf{F}_{2,d}$ as an analogue of the free class-2-nilpotent group on d generators. Lins [20, Cor. 1.5] showed that

$$\mathsf{Z}^{\mathrm{cc}}_{\mathsf{F}_{2,d}}(Y) = \frac{1 - q^{\binom{d-1}{2}}Y}{\left(1 - q^{\binom{d}{2}}Y\right)\left(1 - q^{\binom{d}{2}+1}Y\right)}.$$

Looking back at Example 4.2, we observe that $\mathsf{Z}^{\mathrm{cc}}_{\mathsf{F}_{2,d}}(Y) = \mathsf{Z}^{\mathrm{ask}}_{\mathfrak{so}_d(\mathfrak{O})}(q^{\binom{d}{2}}Y)$; this is no coincidence, see [27, Ex. 7.3].

Example 4.5. Let U_d be the group scheme of upper unitriangular $d \times d$ matrices over $\mathfrak{O}$. Suppose that $\gcd(q, (d-1)!) = 1$. By [26, Thm 1.7] (cf. [7, Prop. 4.12]) and Example 4.3, we have $\mathsf{Z}^{\mathrm{oc}}_{\mathsf{U}_d}(Y) = \frac{(1-Y)^{d-1}}{(1-qY)^d}$.

Graphs and graphical groups. Given a (finite, simple) graph Γ with distinct vertices $v_1, \dots, v_n$ and m edges, let M_Γ be the module of antisymmetric $n \times n$ matrices $A = [a_{ij}]$ such that $a_{ij} = 0$ whenever v_i and v_j are non-adjacent. We write $\mathsf{Z}^{\mathrm{ask}}_{\Gamma}(Y)$ for $\mathsf{Z}^{\mathrm{ask}}_{M(\Gamma)}(Y)$. As shown in [30, Thm A], $\mathsf{Z}^{\mathrm{ask}}_{\Gamma}(Y)$ is a rational function in q and Y. In [30, §3.4], the graphical group scheme G_Γ associated with Γ is constructed; for an alternative but equivalent construction, see [28, §1.1]. By [30, Prop. 3.9], $\mathsf{Z}^{\mathrm{cc}}_{\mathsf{G}_\Gamma}(Y) = \mathsf{Z}^{\mathrm{ask}}_{\Gamma}(q^m Y)$. Given graphs Γ_1 and Γ_2, let $\Gamma_1 \vee \Gamma_2$ denote their join, obtained from the disjoint union of Γ_1 and Γ_2 by adding edges connecting each vertex of Γ_1 to each vertex of Γ_2. Let K_n (resp. Δ_n) denote the complete (resp. edgeless) graph on n vertices.

Example 4.6. Consider $\Gamma = \Delta_n \vee \mathrm{K}_{n+1}$. Then Γ has $3\binom{n+1}{2}$ edges. It follows from [30, Thm 8.18] that

$$\mathsf{Z}^{\mathrm{ask}}_{\Gamma}(Y) = \frac{(1 - q^{-n}Y)(1 - q^{-n-1}Y)}{(1 - q^{-1}Y)(1 - Y)(1 - qY)}.$$

Hadamard products and zeta functions. Let $\mathfrak{O}$ be as above. Elaborating further on what we wrote in the introduction, modules of matrices, (linear) group schemes, and graphs all admit natural operations which correspond to taking Hadamard products of zeta functions. In detail, given modules $M \subseteq \mathrm{M}_{d\times e}(\mathfrak{O})$ and $M' \subseteq \mathrm{M}_{d'\times e'}(\mathfrak{O})$, we regard $M \oplus M'$ as a submodule of $\mathrm{M}_{(d+d')\times(e+e')}(\mathfrak{O})$, embedded diagonally. Then $\mathsf{Z}^{\mathrm{ask}}_{M\oplus M'}(Y) = \mathsf{Z}^{\mathrm{ask}}_{M}(Y) *_Y \mathsf{Z}^{\mathrm{ask}}_{M'}(Y)$. Similarly, given (linear) group schemes G and G' over $\mathfrak{O}$, we obtain $\mathsf{Z}^{\mathrm{cc}}_{\mathsf{G}\times\mathsf{G}'}(Y) = \mathsf{Z}^{\mathrm{cc}}_{\mathsf{G}}(Y) *_Y \mathsf{Z}^{\mathrm{cc}}_{\mathsf{G}'}(Y)$. Finally, given graphs Γ and Γ', let $\Gamma \oplus \Gamma'$ denote their disjoint union. Then $\mathsf{Z}^{\mathrm{ask}}_{\Gamma\oplus\Gamma'}(Y) = \mathsf{Z}^{\mathrm{ask}}_{\Gamma}(Y) *_Y \mathsf{Z}^{\mathrm{ask}}_{\Gamma'}(Y)$; moreover, $\mathsf{G}_{\Gamma\oplus\Gamma'} \cong \mathsf{G}_\Gamma \times \mathsf{G}_{\Gamma'}$.

4.2 Applications

It turns out that each zeta function from Examples 4.1–4.6 can be expressed in terms of the rational functions $W^{e}_{f,\alpha}(X,Y)$ attached to labelled coloured configurations as in Section 2. Omitting proofs, details of this are recorded in Table 1. This table is to be read as follows: for each row and each compact discrete valuation ring $\mathfrak{O}$ with residue field size q (possibly subject to further conditions on $\mathfrak{O}$ as in the examples above), the zeta function $\mathsf{Z}(Y)$ indicated in the leftmost column is obtained from the rational function in the rightmost column via $\mathsf{Z}(Y) = W^{\varepsilon}_{f,\alpha}(q, u(q)Y)$. In Table 1, we write $\underline{n}$ for the sum of all 2^n coloured permutations of the form $1^{\nu_1}\cdots n^{\nu_n}$ with $\nu_i \in \{0, i\}$. A "%" indicates that an entry coincides with the one immediately above it. We note that Table 1 does not constitute an exhaustive list of zeta functions expressible in terms of coloured configurations; we refer to our upcoming paper [6] for further examples and applications.

Zeta function	f	α	ε	$u(X)$	$W^{\varepsilon}_{f,\alpha}(X,Y)$
$\mathsf{Z}^{\mathrm{ask}}_{\mathrm{M}_{d\times e}(\mathfrak{O})}(Y)$	$\underline{1}$	$1 \leftarrow -X^{-d}$	$d-e$	1	$\frac{1-X^{-e}Y}{(1-Y)(1-X^{d-e}Y)}$
$\mathsf{Z}^{\mathrm{ask}}_{\mathfrak{so}_d(\mathfrak{O})}(Y)$, $\mathsf{Z}^{\mathrm{ask}}_{\mathrm{M}_{d\times(d-1)}(\mathfrak{O})}(Y)$	%	%	1	%	$\frac{1-X^{1-d}Y}{(1-Y)(1-XY)}$
$\mathsf{Z}^{\mathrm{cc}}_{\mathsf{F}_{2,d}}(Y)$	%	%	%	$X^{\binom{d}{2}}$	%
$\mathsf{Z}^{\mathrm{ask}}_{\Delta_n \vee \mathrm{K}_{n+1}}(Y)$	$\underline{2}$	$1,2 \leftarrow -X^{-n-1}$	1	X^{-1}	$\frac{(1-X^{1-n}Y)(1-X^{-n}Y)}{(1-Y)(1-XY)(1-X^2Y)}$
$\mathsf{Z}^{\mathrm{cc}}_{\mathsf{G}_{\Delta_n \vee \mathrm{K}_{n+1}}}(Y)$	%	%	%	$X^{3\binom{n+1}{2}-1}$	%
$\mathsf{Z}^{\mathrm{oc}}_{\mathsf{U}_{d+1}}(Y)$	$\underline{d}$	$1,\dots,d \leftarrow -X^{-1}$	0	X	$\frac{(1-X^{-1}Y)^d}{(1-Y)^{d+1}}$

Table 1: Examples of zeta functions from labelled coloured configurations

We now explain how, subject to a compatibility condition, Theorem 2.2 can be used to explicitly compute Hadamard products of the zeta functions in Table 1. As explained in Section 4.1, we can interpret such Hadamard products as zeta functions associated with "products" of the objects under consideration. We first record an elementary fact.

Lemma 4.7. *Let R be a commutative ring. Let $A(Y) = \sum_{k=0}^{\infty} a_k Y^k$ and $B(Y) = \sum_{k=0}^{\infty} b_k Y^k$ be formal power series over R. Let $u, v \in R$. Then $A(uY) *_Y B(vY) = (A *_Y B)(uvY)$.* □

The compatibility condition that we alluded to above is that we require entries in the ε-column of Table 1 to agree for us to compute associated Hadamard products via Theorem 2.2 and Lemma 4.7. Thus, suppose that (f,α) and (g,β) are coloured configurations and let $u(X)$ and $v(X)$ each be of the form $\pm X^{\ell}$ for $\ell \in \mathbb{Z}$. Then

$$W^{\varepsilon}_{f,\alpha}\big(X,\, u(X)Y\big) *_Y W^{\varepsilon}_{g,\beta}\big(X,\, v(X)Y\big) = (W^{\varepsilon}_{f,\alpha} *_Y W^{\varepsilon}_{g,\beta})\big(X,\, u(X)v(X)Y\big).$$

As explained in Section 2, by passing to equivalent labelled coloured configurations, we may assume that f and g are strongly disjoint. Theorem 2.2 then allows us to explicitly compute $W^{\varepsilon}_{f,\alpha} *_Y W^{\varepsilon}_{g,\beta}$. (Here it is crucial that a common value of ε is used in both factors.) All that remains to obtain our zeta function is to apply the specialisation $X \leftarrow q$.

To illustrate the scope of our method by means of, say, a group-theoretic application, first note that for *specific* choices of $d_1, \dots, d_r$, a finite computation (using the algorithm in [30, §6], implemented in the software package Zeta [29]) can be used to determine $\mathsf{Z}^{\mathsf{cc}}_{\mathsf{F}_{2,d_1} \times \dots \times \mathsf{F}_{2,d_r}}(Y)$. Our results here go substantially further. Indeed, for any fixed r, an explicit finite computation produces a single formula for $\mathsf{Z}^{\mathsf{cc}}_{\mathsf{F}_{2,d_1} \times \dots \times \mathsf{F}_{2,d_r}}(Y) = \mathsf{Z}^{\mathsf{cc}}_{\mathsf{F}_{2,d_1}}(Y) *_Y \dots *_Y \mathsf{Z}^{\mathsf{cc}}_{\mathsf{F}_{2,d_r}}(Y)$ as a *symbolic expression* involving variables $d_1, \dots, d_r$.

Example 4.8. By combining Example 2.3 and Example 4.4, we find that $\mathsf{Z}^{\mathsf{cc}}_{\mathsf{F}_{2,d} \times \mathsf{F}_{2,d'}}(Y) = W\big(q, q^{\binom{d}{2}+\binom{d'}{2}} Y\big)$, where

$$W(X,Y) = \frac{1 + (1 - X^{-d} - X^{-d'})XY + (X^{-d-d'} - X^{-d} - X^{-d'})X^2Y + X^{-d-d'}X^3Y^2}{(1-Y)(1-XY)(1-X^2Y)}.$$

In the same spirit, using the data from Table 1, we can e.g. symbolically compute the orbit-counting zeta function of $\mathsf{U}_{d_1} \times \dots \times \mathsf{U}_{d_r}$ and the ask zeta function of $\mathrm{M}_{d_1 \times e_1}(\mathfrak{O}) \oplus \dots \oplus \mathrm{M}_{d_r \times e_r}(\mathfrak{O})$ when $d_1 - e_1 = \dots = d_r - e_r$, all for fixed r but symbolic variables $d_1, \dots, d_r$ and $e_1, \dots, e_r$. In particular, the latter case provides an explicit description of the ask zeta function of $\mathrm{M}_{d_1}(\mathfrak{O}) \oplus \dots \oplus \mathrm{M}_{d_r}(\mathfrak{O})$ in terms of coloured permutation statistics; cf. [30, Question 10.4(a)–(b)]. Such a description was previously only known for $d_1 = \dots = d_r$; see [30, Prop. 10.3].

We conclude by noting that, to the best of our knowledge, the previously mentioned result [30, Prop. 10.3] and its close relative [26, Cor. 5.17] (both pertaining to direct powers of $\mathrm{M}_d(\mathfrak{O})$) were the only examples of Hadamard products of ask (as well as class- and orbit-counting) zeta functions expressed in terms of (coloured) permutation statistics. In the aforementioned results in the literature, the proofs rely on work of Brenti [3]. These proofs are based on the coincidence of the rational generating functions in question with those attached to so-called q-Eulerian polynomials of signed permutations. This preceded more recent machinery surrounding shuffle compatibility. This earlier work is now explained as part of the framework presented here.

Acknowledgements

AC and TR gratefully acknowledge interesting discussions with Christopher Voll.

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