

The intermediate orders of a Coxeter group

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Abstract

We define a class of partial orders on a Coxeter group associated with sets of reflections. In special cases, these lie between the left weak order and the Bruhat order. We prove that these posets are graded by the length function and that the projections on the right parabolic quotients are always order preserving. We also introduce the notion of k -Bruhat graph, k -absolute length and k -absolute order, proposing some related conjectures and problems.

1 Introduction

The weak order and the Bruhat order of a Coxeter group are partial orders of preeminent importance in wide parts of algebraic combinatorics, representation theory and algebraic geometry. These orders depend on the Coxeter presentation of the group and the Bruhat order is a refinement of the weak order, once a presentation is chosen. Both orders are graded by the length function of the group. The weak order is a complete meet-semilattice (and an orthocomplemented lattice in the finite case) and the order complex of its open intervals are homotopy equivalent to spheres or are contractible. On the other hand, the Bruhat order is Eulerian and its open intervals are shellable. See [3, Ch. 2 and 3] and references therein for these and other properties.

In this article we introduce a new class of partial orders on any Coxeter group associated with sets of reflections. For the sake of simplicity, we illustrate here a special case of particular interest. For each $k \in \mathbb{N}$ we define an order \leq_{L^k} on a Coxeter group W associated with the set of reflections

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whose length is bounded (in a way that depends on k). If $a < b$ then (W, \leq_{L^b}) is a refinement of (W, \leq_{L^a}) , for all $a, b \in \mathbb{N}$. Moreover, $\leq_{L^0} = \leq_L$ is the left weak order on W . If \leq is the Bruhat order and $k \in \mathbb{N}$, we obtain a sequence of injective poset morphisms

$$(W, \leq_L) \hookrightarrow (W, \leq_{L^1}) \hookrightarrow \dots \hookrightarrow (W, \leq_{L^k}) \hookrightarrow (W, \leq).$$

For this reason we call the new orders in this special case *k-intermediate orders*; cf. Definition 3.1.

Our first main result pertaining to these orders is the following. We will prove it as a consequence of a more general result; see Theorem 3.3 and Corollary 3.5.

Theorem. *The poset (W, \leq_{L^k}) is graded by the Coxeter length for all $k \in \mathbb{N}$.*

Consider the function $P^J : W \rightarrow W$ which assigns to an element $w \in W$ the representative of minimal length of the coset wW_J , where $W_J \subseteq W$ is the parabolic subgroup generated by J . Our second main result for k -intermediate orders is the following (cf. Theorem 3.7).

Theorem. *The functions P^J are order preserving on (W, \leq_{L^k}) for all $k \in \mathbb{N}$.*

After setting up some notation and recalling some preliminaries in Section 2, we will prove these theorems as consequences of more general results pertaining to a broader family of posets; see Section 3.

Our k -intermediate orders are defined by considering sets of reflections with bounded length, as the Bruhat order is defined by considering the whole set of reflections. In the same spirit, we define in Section 4 various other k -analogues of related objects. In particular, we define the *k-absolute orders* (see Definition 4.1). Both the k -intermediate and k -absolute orders coincide with the weak order if $k = 0$. The absolute order of the symmetric group was introduced by T. Brady in [4] and it is involved in the construction of Eilenberg-MacLane spaces for the braid groups. He also proved that the lattice of noncrossing partitions NC_n is isomorphic to any of the maximal intervals in the absolute order of S_n corresponding to Coxeter elements. The absolute order on a Coxeter group has been subsequently considered in several kinds of problems concerning shellability, Cohen-Macaulay, Sperner and spectral properties, among others; see e.g. [1], [2], [10], [11], [13]. Inspired by these works we formulate some problems and conjectures about k -intermediate orders, k -absolute orders and their rank function, which we call *k-absolute length*.

We also introduce the k -Bruhat graph of a Coxeter system, which turns out to be always locally finite and in some sense approximates the Bruhat

graph when the group is infinite. Recent results on the Ricci curvature of Bruhat and Cayley graphs of Coxeter groups ([15], [16]) lead to consider the Ricci curvature of a k -Bruhat graph; this problem closes the paper.

2 Notation and preliminaries

In this section we establish some notation and we collect some basic results from the theory of Coxeter systems which are useful in the sequel. The reader can consult [3] and references therein for further details. We follow [18, Ch. 3] for notation and terminology concerning posets.

With \mathbb{N} we denote the set of non-negative integers. For $n \in \mathbb{N}$ we let $[n] := \{1, 2, \dots, n\}$. With \uplus we denote the disjoint union and with $|X|$ the cardinality of a set X . Given any category, $\text{End}(O)$ denotes the set of endomorphisms of an object O .

Let (W, S) be a Coxeter system. That is, W is a group with a presentation given by a finite set of involutive generators S and relations encoded by a *Coxeter matrix* $m : S \times S \rightarrow \{1, 2, \dots, \infty\}$ (see [3, Ch. 1]). A Coxeter matrix over S is a symmetric matrix which satisfies the following conditions for all $s, t \in S$:

1. $m(s, t) = 1$ if and only if $s = t$;
2. $m(s, t) \in \{2, 3, \dots, \infty\}$ if $s \neq t$.

The presentation (W, S) of the group W is then the following:

$$\begin{cases} \text{generators : } & S; \\ \text{relations : } & (st)^{m(s,t)} = e, \end{cases}$$

for all $s, t \in S$, where e denotes the identity in W . The Coxeter matrix m attains the value ∞ at (s, t) to indicate that there is no relation between the generators s and t . The class of words expressing an element of W contains words of minimal length; the *length function* $\ell : W \rightarrow \mathbb{N}$ assigns to an element $w \in W$ such minimal length. The identity e is represented by the empty word and then $\ell(e) = 0$. A *reduced word* or *reduced expression* for an element $w \in W$ is a word of minimal length representing w . The set of *reflections* of (W, S) is defined by $T := \{wsw^{-1} : w \in W, s \in S\}$. If $J \subseteq S$ and $v \in W$, we let

$$\begin{aligned} W^J &:= \{w \in W : \ell(w) < \ell(ws) \forall s \in J\}, \\ {}^J W &:= \{w \in W : \ell(w) < \ell(sw) \forall s \in J\}, \\ D_L(v) &:= \{s \in S : \ell(sv) < \ell(v)\}, \\ D_R(v) &:= \{s \in S : \ell(vs) < \ell(v)\}. \end{aligned}$$

With W_J we denote the subgroup of W generated by $J \subseteq S$; such a group is usually called a *parabolic subgroup* of W . In particular, $W_S = W$ and $W_\emptyset = \{e\}$.

Given a Coxeter system (W, S) , we let \leq_L and \leq be the *left weak order* and the *Bruhat order* on W , respectively. The covering relations of the left weak order are characterized as follows: $u \triangleleft v$ if and only if $\ell(u) < \ell(v)$ and $uv^{-1} \in S$. The covering relations of the Bruhat order are characterized as follows: $u \triangleleft v$ if and only if $\ell(u) = \ell(v) - 1$ and $uv^{-1} \in T$. The posets (W, \leq_L) and (W, \leq) are graded with rank function ℓ and $(W, \leq_L) \hookrightarrow (W, \leq)$. We recall a characterizing property of the Bruhat order, known as *lifting property* (see [3, Proposition 2.2.7]):

Proposition 2.1 (Lifting Property). *Let $v, w \in W$ such that $v < w$ and $s \in D_L(w) \setminus D_L(v)$. Then $v \leq sw$ and $sv \leq w$.*

For $J \subseteq S$, each element $w \in W$ factorizes uniquely as $w = w^J w_J$, where $w^J \in W^J$, $w_J \in W_J$ and $\ell(w) = \ell(w_J) + \ell(w^J)$; see [3, Proposition 2.4.4]. We consider the idempotent function $P^J : W \rightarrow W$ defined by

$$P^J(w) = w^J,$$

for all $w \in W$. This function is order preserving for the Bruhat order (see [3, Proposition 2.5.1]). In the next section we prove that the function P^J is order preserving for a wider class of partial orders which we are going to introduce. In a similar way, one defines an order-preserving function $Q^J : (W, \leq) \rightarrow (W, \leq)$ by setting $Q^J(w) = {}^J w$, where $w = w'_J {}^J w$ with $w'_J \in W_J$, ${}^J w \in {}^J W$ and $\ell(w) = \ell(w'_J) + \ell({}^J w)$.

We end this section by recalling a theorem about the Bruhat graph of a Coxeter system (W, S) and the reflection subgroups of W . The *Bruhat graph* of (W, S) is the directed graph $\Omega_{W,S}$ whose vertex set is W and such that there is an arrow from u to v if and only if $uv^{-1} \in T$ and $u < v$; such an arrow is labeled by the reflection uv^{-1} . If $Y \subseteq W$, we denote by $\Omega_{W,S}(Y)$ the induced subgraph of $\Omega_{W,S}$ whose vertex set is Y . A subgroup $W' \subseteq W$ generated by $X \subseteq T$ is called a *reflection subgroup* of W and (W', S') is a Coxeter system, where $S' := \{t \in T : N(t) \cap W' = \{t\}\}$ and $N(v) := \{t \in T : \ell(tv) < \ell(v)\}$; see [7]. The following theorem is [8, Theorem 1.4].

Theorem 2.2. *Let (W, S) be a Coxeter system and $W' \subseteq W$ a reflection subgroup with set of Coxeter generators S' . Then*

1. $\Omega_{W',S'} = \Omega_{W,S}(W')$;
2. for all $x \in W$ there exists $x_0 \in W'x$ such that the function $W' \rightarrow W'x$ given by $w \mapsto wx_0$ induces an isomorphism of directed graphs between $\Omega_{W',S'}$ and $\Omega_{W,S}(W'x)$ which preserves the labels of the edges.

3 Intermediate orders

In this section we introduce the main objects of our study. Let (W, S) be a Coxeter system and T its set of reflections. For $k \in \mathbb{N}$ we let

$$T_k := \left\{ t \in T : \frac{\ell(t) - 1}{2} \leq k \right\}.$$

The following definition introduces the notion of k -intermediate order.

Definition 3.1. *Let $X \subseteq T$. We define a partial order \leq_{L^X} on W by letting $u \leq_{L^X} v$ if and only if*

- $u = v$ or
- $\ell(u) < \ell(v)$ and there exist $t_1, \dots, t_r \in X$ such that

$$u < t_1 u < t_2 t_1 u < \dots < t_r \dots t_1 u = v.$$

For $k \in \mathbb{N}$ and $X = T_k$, we denote \leq_{L^X} by \leq_{L^k} and call it a k -intermediate order on W .

Note that $(W, \leq_{L^0}) = (W, \leq_L)$ and, if W is finite, for k big enough $(W, \leq_{L^k}) = (W, \leq)$. We have that $(W, \leq_{L^0}) \hookrightarrow (W, \leq_{L^1}) \hookrightarrow \dots \hookrightarrow (W, \leq)$, which justifies the name ‘ k -intermediate orders’. For $u, v \in W$ such that $u \leq_{L^k} v$, we denote by $[u, v]_k$ the corresponding interval in (W, \leq_{L^k}) and by $[u, v]$ the interval in (W, \leq) .

To ease the notation, we write \triangleleft_X for a covering relation in (W, \leq_{L^X}) .

Example 3.2. *Let $(W, S) = (S_4, \{s_1, s_2, s_3\})$, the symmetric group of order 24 with its standard Coxeter presentation. Then $(S_4, \leq_{L^2}) = (S_4, \leq)$, whose Hasse diagram is displayed in [3, Figure 2.4]. The Hasse diagram of the poset $(S_4, \leq_{L^0}) = (S_4, \leq_L)$, i.e. the Cayley graph of (S_4, S) , appears in [3, Figure 3.2]. The poset (S_4, \leq_{L^1}) is depicted in Figure 1.*

In order to prove the first main result of this paper, we define a poset (T, \sqsubseteq) as follows. Let $\Omega_{W,S}$ be the Bruhat graph of (W, S) ; we define a relation \sqsubseteq' on T by setting $t \sqsubseteq' t'$ if and only if $t = t'$ or there exists a reflection subgroup $W' \subseteq W$ such that:

1. $t, t' \in W'$;
2. the system (W', S') is dihedral, i.e. $|S'| = 2$;
3. the distance $\vec{d}(t, t')$ in $\Omega_{W,S}(W') = \Omega_{W',S'}$ is finite.

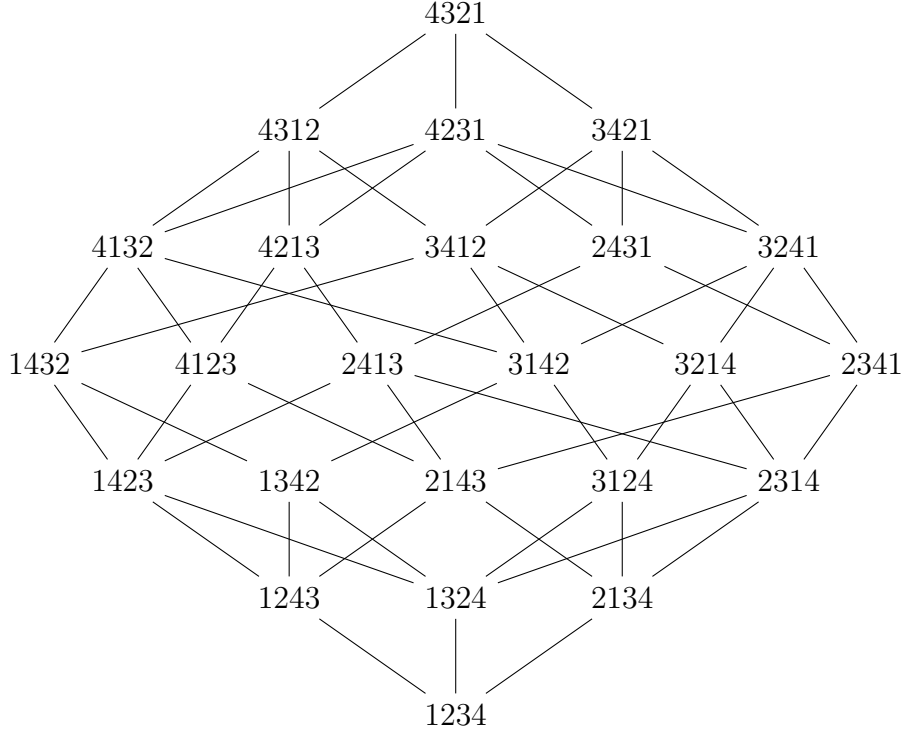


Figure 1: Hasse diagram of (S_4, \leq_{L^1}) .

We define \sqsubseteq as the transitive closure of the relation \sqsubseteq' . By definition, $(T, \sqsubseteq) \hookrightarrow (T, \leq)$ as posets, where \leq is the Bruhat order. The Hasse diagrams of (T, \sqsubseteq) in types A_3 and B_3 are displayed in Figure 2.

We now prove our first theorem.

Theorem 3.3. *Let (W, S) be a Coxeter system and $X \subseteq T$ be an order ideal of (T, \sqsubseteq) . Then the poset (W, \leq_{L^X}) is graded with rank function $\ell - \ell \circ Q^{S \cap X}$.*

Proof. First we prove that, if $X \subseteq T$ is an order ideal of (T, \sqsubseteq) and $u \triangleleft_X v$, then $\ell(v) - \ell(u) = 1$, for all $u, v \in W$. Let $u \triangleleft_X v$; then $\ell(u) < \ell(v)$ and there exists $t \in X$ such that $u = tv$. Assume $\ell(u) < \ell(v) - 1$. Then there exists a dihedral reflection subgroup $W' \subseteq W$ such that $t \in W'$, and a path from v to u in the directed graph $\Omega_{W,S}(W'u) = \Omega_{W,S}(W'v)$, whose length is strictly greater than 1. This is Claim (i) in the proof of [8, Proposition 3.3]. Moreover, if (W, S) is a dihedral Coxeter system, $x \in W$ and $t \in T$ such that $\ell(tx) > \ell(x) + 1$, then there exist $t_1, \dots, t_n \in T$ satisfying $x < t_1x < \dots < t_n \cdots t_1x = tx$ and $\ell(t_i) < \ell(t)$ for all $1 \leq i \leq n$. Notice that, in a dihedral Coxeter system, $\ell(t') < \ell(t)$ if and only if $t' \sqsubset t$, for any $t', t \in T$. Hence there exists a chain $u =: x_0 \triangleleft x_1 \triangleleft \dots \triangleleft x_n := v$ in (W, \leq) such that $\ell(x_{i-1}x_i^{-1}) < \ell(t)$, for all

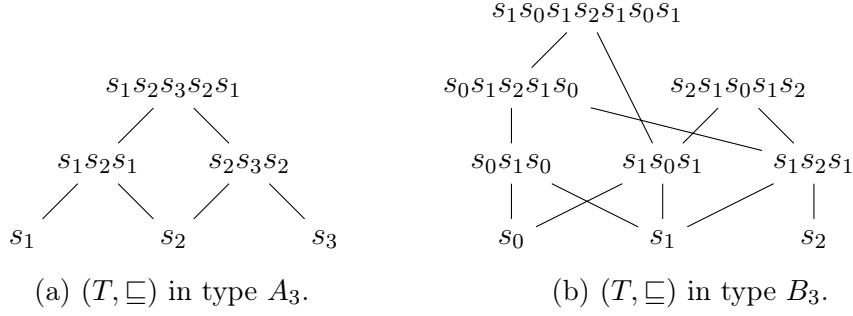


Figure 2: Examples of (T, \sqsubseteq) .

$i \in [n]$; then $x_{i-1}x_i^{-1} \in X$ for all $i \in [n]$, since X is an order ideal of (T, \sqsubseteq) . Therefore $u \triangleleft_X v$ implies $\ell(v) - \ell(u) = 1$.

Let $J := S \cap X$. If $S \subseteq X$ we have that $J = S$, $(\ell \circ Q^J)(w) = 0$ for all $w \in W$ and, as posets, $(W, \leq_L) \leftrightarrow (W, \leq_{L^X})$; therefore, in this case, the rank function of (W, \leq_{L^X}) is ℓ . Let us consider the case $S \not\subseteq X$. First note that $s < t \in X$ implies $s \in J$, for all $s \in S$. To see this, first note that t has a palindromic reduced expression (see [8, Lemma 2.7]). Since $s < t$, either $t = wsw^{-1}$ with $\ell(wsw^{-1}) = 2\ell(w) + 1$ for some $w \in W$, or $t = wsrsw^{-1}$ for some $w \in W$ and $r \in T$ with $\ell(t) = 2\ell(w) + 2 + \ell(r)$. By induction on $\ell(w)$, it follows that $s \sqsubseteq t$ in the first case, and $srs \sqsubseteq t$ in the second. We claim $s \sqsubseteq t$ in the second case also. For let $W' := \langle s, r \rangle$; therefore $s, srs \in W'$ and $s \rightarrow rs \rightarrow srs$ is a path in the directed graph $\Omega_{W,S}(W')$, and so $s \sqsubseteq srs \sqsubseteq t$. Therefore $s < t \in X$ implies $s \in X$ and then $s \in J$. So we have that $X \subseteq W_J$ and X is an order ideal of $(W_J \cap T, \sqsubseteq)$. Since $X \subseteq W_J$, there is a poset isomorphism $(W_J w, \leq_{L^X}) \simeq (W_J, \leq_{L^X})$, for all $w \in {}^J W$. Then we conclude using the fact that $W = \bigoplus_{w \in {}^J W} W_J w$. \square

Remark 3.4. In the proof of Theorem 3.3 we see that, if $X \subseteq T$ is an order ideal of (T, \sqsubseteq) , then

$$(W, \leq_{L^X}) \simeq \bigoplus_{i=1}^{|{}^J W|} (W_J, \leq_{L^X}),$$

the coproduct of $|{}^J W|$ copies of the poset (W_J, \leq_{L^X}) , where $J := S \cap X$.

Clearly, the set T_k is an order ideal of (T, \leq) , and then of (T, \sqsubseteq) ; moreover $S \subseteq T_k$. These observations imply the following corollary.

Corollary 3.5. *Let $k \in \mathbb{N}$. Then the poset (W, \leq_{L^k}) is graded with rank function ℓ .*

Note that if $X \supseteq S$ is an order ideal of T , then (W, \leq_{L^X}) is graded with rank function ℓ , and $(W, \leq_L) \hookrightarrow (W, \leq_{L^X}) \hookrightarrow (W, \leq)$.

A finite graded poset is *strongly Sperner* if no union of h antichains is larger than the union of the h largest rank levels, for all $h \in \mathbb{N}$. By a recent result of Gaetz and Gao [9], the poset (S_n, \leq_L) is strongly Sperner. Since under the hypotheses below any antichain of (W, \leq_{L^X}) is an antichain of (W, \leq_L) , the following result holds for symmetric groups.

Corollary 3.6. *Let X be an order ideal of T containing S . Then the poset (S_n, \leq_{L^X}) is strongly Sperner. In particular, the poset (S_n, \leq_{L^k}) is strongly Sperner if $n > 0$ and $k \in \mathbb{N}$.*

The category of posets considered here is the one with posets as objects and order-preserving functions as morphisms. Let (W, S) be a Coxeter system and $J \subseteq S$. Then the function $Q^J : W \rightarrow W$ is not, in general, an element of $\text{End}(W, \leq_{L^X})$. For example, in type A_2 with Coxeter generators $\{s, t\}$, we have $ts \leq_L sts$ but $Q^{\{t\}}(ts) = s \not\leq_L st = Q^{\{t\}}(sts)$. The second main result of this article is that, in the same hypotheses as Theorem 3.3, the functions $P^J : (W, \leq_{L^X}) \rightarrow (W, \leq_{L^X})$ are order preserving.

Theorem 3.7. *Let (W, S) be a Coxeter system, $J \subseteq S$ and $X \subseteq T$ be an order ideal of (T, \sqsubseteq) . Then $P^J \in \text{End}(W, \leq_{L^X})$.*

Proof. We first prove the result in the case $|J| = 1$. Let $J = \{s\}$ for some $s \in S$, and let $u \triangleleft_X v$. Then there exists $t \in X$ such that $u = tv$ and, by Theorem 3.3, $\ell(v) - \ell(u) = 1$. We proceed by induction on $\ell(v)$. If $\ell(v) = 1$ then $u = e$ and the result is obvious. Let $\ell(v) > 1$. We consider two cases.

1. Suppose $vs < v$. If $u < us$ then $us = v$ and so $P^J(u) = u = P^J(v)$. If $us < u$ then $us = tvs$ and $\ell(tvs) = \ell(vs) - 1$; hence $P^J(u) = us \triangleleft_X vs = P^J(v)$.
2. Suppose now $v < vs$. If $u_J = e$ then $P^J(u) = u \triangleleft_X v = P^J(v)$. If instead $u_J = s$ then $u^J = tvs$ and $\ell(u^J) = \ell(vs) - 3$. Hence $u^J \leq_{L^X} vs$. By Theorem 3.3 there exist $w_1, w_2 \in W$ such that $u^J \triangleleft_X w_1 \triangleleft_X w_2 \triangleleft_X vs$. If $w_2 < w_2s$ then $w_2 = v = v^J$ and the result follows. So let $w_2s < w_2$. Since $\ell(w_2) = \ell(v)$, by case 1 above and our inductive hypothesis $u^J \leq_{L^X} P^J(w_1) \leq_{L^X} P^J(w_2)$. Moreover, $P^J(w_2) = w_2s = (rvs)s = rv$, for some $r \in X$, and $\ell(w_2s) = \ell(v) - 1$. Hence $u^J \leq_{L^X} P^J(w_1) \leq_{L^X} P^J(w_2) \triangleleft_X v$.

Let $u \leq_{L^X} v$ such that $\ell(u) < \ell(v) - 1$. Then, by Theorem 3.3 there exists a chain $u \triangleleft_X w_1 \triangleleft_X \dots \triangleleft_X w_n \triangleleft_X v$. Therefore

$$P^J(u) \leq_{L^X} P^J(w_1) \leq_{L^X} \dots \leq_{L^X} P^J(w_n) \leq_{L^X} P^J(v).$$

Let now $|J| > 1$ and $u \triangleleft_X v$. We proceed by induction on $\ell(v)$. If $\ell(v) = 1$ then $u = e$ and the result is obvious. Let $\ell(v) > 1$ and consider the following two cases.

1. Suppose $v_J > e$. Let $s \in D_R(v_J)$. If $u < us$ we have that $us = v$ and then $P^J(u) = P^J(us) = P^J(v)$. If $us < u$, as before, $us \triangleleft_X vs$. By our inductive hypothesis $P^J(u) = P^J(us) \leq_{L^X} P^J(vs) = P^J(v)$.
2. Suppose now $v_J = e$. If $u_J = e$, then $P^J(u) = u \triangleleft_X v = P^J(v)$. If $u_J > e$, let $s_1 \cdots s_m$ be a reduced word for u_J . Then $\{s_1, s_2, \dots, s_m\} \subseteq J$ and the result follows from the case $|J| = 1$, since

$$u^J = P^{\{s_1\}} \cdots P^{\{s_m\}}(u^J u_J) \leq_{L^X} P^{\{s_1\}} \cdots P^{\{s_m\}}(v) = v. \quad \square$$

By the same arguments as before, the previous result holds in particular for the orders (W, \leq_{L^k}) , thus proving the second theorem in Section 1.

Remark 3.8. The previous result is known for the Bruhat order (see [3, Proposition 2.5.1]); in that case it is a direct consequence of the lifting property. For $X = S$ the result, in its right version, is [6, Lemma 2.1].

Recall that given a Coxeter system (W, S) with Coxeter matrix m , one can define an associated Coxeter monoid W^m as

$$W^m = \left\langle s_i \in S : s_i^2 = s_i, \underbrace{s_i s_j s_i \cdots}_{m(s_i, s_j) \text{ terms}} = \underbrace{s_j s_i s_j \cdots}_{m(s_i, s_j) \text{ terms}} \right\rangle,$$

see also [12]. It is known that the Coxeter monoid W^m is a submonoid of the monoid M generated by the functions $\{P^J : J \subseteq S\}$; see e.g. [14, Section 4] for details. By Theorem 3.7, M is a submonoid of $\text{End}(W, \leq_{L^k})$. Hence we have the following corollary.

Corollary 3.9. *Let $k \in \mathbb{N}$. Then, as monoids, $W^m \hookrightarrow \text{End}(W, \leq_{L^k})$.*

Let $k \in \mathbb{N}$, $(W, \{s_1, \dots, s_n\})$ be a Coxeter system and let

$$\phi_k : (W, \leq_{L^k}) \rightarrow (W^{S \setminus \{s_1\}}, \leq_{L^k}) \times \cdots \times (W^{S \setminus \{s_n\}}, \leq_{L^k})$$

be the function defined by $\phi_k(w) = (P^{S \setminus \{s_1\}}(w), \dots, P^{S \setminus \{s_n\}}(w))$, for all $w \in W$. By Theorem 3.7 the function ϕ_k is order preserving. Given a function $f : (A, \leq_A) \rightarrow (B, \leq_B)$, we define an induced subposet of B by $\text{Im}(f) := (\{f(a) : a \in A\}, \leq_B)$.

Proposition 3.10. *Let $W = S_n$ with its standard Coxeter presentation. Then $\text{Im}(\phi_k) \simeq (S_n, \leq)$, for all $k \in \mathbb{N}$.*

Proof. It suffices to prove the result for $k = 0$. By [3, Exercise 3.2],

$$(S_n^{(1)}, \leq_L) \times \dots \times (S_n^{(n-1)}, \leq_L) = (S_n^{(1)}, \leq) \times \dots \times (S_n^{(n-1)}, \leq),$$

where, for $h \in [n - 1]$, we have defined $S_n^{(h)} := S_n^{S \setminus \{s_h\}}$. Since $u \leq v$ if and only if $P^{S \setminus \{s_h\}}(u) \leq P^{S \setminus \{s_h\}}(v)$ for all $h \in [n - 1]$ (see, e.g. [3, Theorem 2.6.1]), the result follows. \square

The previous result could be not true for other Coxeter groups. For example, in type B_3 , the poset $\text{Im}(\phi_0)$ is not graded.

4 The k -absolute orders and open problems

In this section we define k -analogues of Bruhat graphs, absolute orders and absolute length. We then formulate a few related conjectures and open problems. These definitions could be extended to orders \leq_{L^X} where X is an order ideal of T containing S , but the conjectures are not stated in that extended setting since they are inadequately tested in that generality.

Let (W, S) be a Coxeter system, $k \geq 0$ and Ω^k the directed graph whose vertex set is W and such that there is an arrow from a to b if and only if $ab^{-1} \in T_k$ and $a < b$. We call Ω^k the k -Bruhat graph of (W, S) . Let $\vec{d}_k(a, b)$ be the distance from a to b in the directed graph Ω^k . We define the k -absolute length of $w \in W$ by

$$\ell_k(w) := \vec{d}_k(e, w).$$

Clearly $\ell_0 = \ell$ and, in the finite case for k big enough, $\ell_k = al$, where al is the absolute length (see, for instance, [3, Exercise 7.2] and references therein).

Definition 4.1. Let $k \in \mathbb{N}$. The left k -absolute order \preceq_k on W is the partial order defined by letting $u \preceq_k v$ if and only if $\ell_k(v) = \ell_k(u) + \ell_k(vu^{-1})$, for all $u, v \in W$.

By definition, the poset (W, \preceq_k) is ranked with rank function ℓ_k . For $k = 0$ we recover the left weak order; in the finite case, for k big enough we obtain the absolute order \preceq first introduced in [4].

Remark 4.2. Notice that the maximal chains of (W, \preceq_k) could have different lengths. For example, if $k = 1$ and $W = S_4$ with its standard Coxeter presentation, then $\max_{\preceq_k} W = \{2413, 3142, 4321\}$, $\ell_1(2413) = \ell_1(s_1 s_3 s_2) = 3$, $\ell_1(3142) = \ell_1(s_2 s_3 s_1) = 3$ and $\ell_1(4321) = \ell_1(s_1 s_2 s_3 t) = 4$, where $t := s_1 s_2 s_1$.

We denote by $[u, v]_a$ an interval in the absolute order and by $[u, v]_{ak}$ an interval in the k -absolute order. Brady proved in [4] that, in type A_n , if c is

any Coxeter element then the interval $[e, c]_a$ is isomorphic to the lattice of noncrossing partitions. In any finite type, the intervals $[e, c]_a$ have been proved to be shellable in [1]. In this vein, we formulate the following conjecture for k -intermediate orders and k -absolute orders.

Conjecture 4.3. *Let c be a Coxeter element of a Coxeter system (W, S) . Then the order complexes of the intervals $[e, c]_k$ and $[e, c]_{ak}$ are shellable, for all $k \in \mathbb{N}$.*

Since Bruhat intervals are shellable, the conjecture is true for Bruhat intervals $[e, c]$. By SageMath computations, we have verified the conjecture for the intervals $[e, c]_k$ in type A_n , for all $1 \leq n \leq 5$, in type B_n , for all $2 \leq n \leq 5$, in types D_4, D_5, F_4, H_3 and H_4 , for all $k \in \mathbb{N}$. We have verified the conjecture for the intervals $[e, c]_{ak}$ in type A_n , for all $1 \leq n \leq 5$, in type B_n , for all $2 \leq n \leq 4$, in types D_4 and H_3 , for all $k \in \mathbb{N}$.

Another feature of an absolute order is its strong Sperner property. It has been proved in [11], using flow techniques on Hasse diagrams, that the poset (S_n, \preceq) is strongly Sperner. This result has been stated in [10] for finite Coxeter groups, except for type D_n . Hence we can propose the following problem for finite Coxeter groups.

Problem 4.4. *Study the Sperner properties of (W, \preceq_k) .*

In the previous section we mentioned that (S_n, \preceq_0) is strongly Sperner and from this fact we could deduce Corollary 3.6. For other finite Coxeter groups and $k = 0$ the strong Sperner property is expected in [9, Conjecture 3.1].

We now consider the distribution of ℓ_k on any Coxeter group.

Example 4.5. *The generating function of ℓ_1 on S_4 is*

$$\sum_{w \in S_4} x^{\ell_1(w)} = 1 + 5x + 10x^2 + 7x^3 + x^4.$$

Remark 4.6. The coefficient of x in $\sum_{w \in W} x^{\ell_k(w)}$ is the number of reflections in T_k . For $W = S_n$, this turns out to be $|T_k| = \binom{n}{2} - \binom{n-k-1}{2}$. Indeed, it is easy to see that the number of transpositions in $T \setminus T_k$, that is the number of transpositions with length strictly greater than $2k + 1$, is $\binom{n-k-1}{2}$.

For definitions and results about log-concavity and unimodality we refer to [17] and [5]. We put forward two conjectures about the generating functions of the k -absolute length on finite Coxeter systems. The first is as follows.

Conjecture 4.7. *Let (W, S) be a finite Coxeter system and $k \geq 0$. Then the polynomial $\sum_{w \in W} x^{\ell_k(w)}$ is log-concave with no internal zeros.*

By [17, Proposition 2] and the known factorization of the polynomials $\sum_{w \in W} x^{\ell(w)}$ and $\sum_{w \in W} x^{al(w)}$ (see, e.g. [3, Theorem 7.1.5] and [3, Exercise 7.2]), the previous conjecture holds in these two extreme cases.

For dihedral groups $I_2(m)$, the statement of Conjecture 4.7 holds. Indeed, the following formula can be directly verified. Let $h \in \mathbb{N} \setminus \{0\}$ and define the function $\pi_h : \mathbb{N} \rightarrow \mathbb{N}$ by $\pi_h(n) = n - h \lfloor n/h \rfloor$, for all $n \in \mathbb{N}$. If $1 \leq 2k+1 \leq m$, then

$$\sum_{w \in I_2(m)} x^{\ell_k(w)} = 1 + 2(k+1)x + 2(2k+1) \sum_{i=2}^{\lfloor \frac{m-1}{2k+1} \rfloor} x^i + a_{k,m} x^{\lfloor \frac{m-1}{2k+1} \rfloor + 1} + b_{k,m} x^{\lfloor \frac{m-1}{2k+1} \rfloor + 2},$$

where

$$a_{k,m} := 2k + \pi_{2k+1}(m) + \begin{cases} 2k + 1, & \text{if } \pi_{2k+1}(m) = 0; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$b_{k,m} := \begin{cases} 2k, & \text{if } \pi_{2k+1}(m) = 0; \\ \pi_{2k+1}(m) - 1, & \text{otherwise.} \end{cases}$$

Explicit computations carried out with SageMath [19] confirm Conjecture 4.7 for all relevant k for types A_n ($n \leq 5$), B_n ($n \leq 4$), D_n ($n \leq 5$), F_4 and H_3 . Since a non-negative log-concave sequence with no internal zeros is unimodal, the previous conjecture implies the following.

Conjecture 4.8. *Let (W, S) be a finite Coxeter system and $k \geq 0$. Then the polynomial $\sum_{w \in W} x^{\ell_k(w)}$ is unimodal.*

We end this section with a problem on the Ricci curvature of k -Bruhat graphs. The Ricci curvature of the graph Ω^0 (the Cayley graph of (W, S)) is studied (and in many cases explicitly computed) in [15]; the Ricci curvature of the Bruhat graph of a finite Coxeter group is proved to be 2 in [16]. We refer to these articles for definitions and preliminary results. Note that a k -Bruhat graph is always locally-finite and is a subgraph of the Bruhat graph; it is then natural to formulate the following problem.

Problem 4.9. *Let (W, S) be a Coxeter system and $k \geq 0$. What can be said about the Ricci curvature of the k -Bruhat graph?*

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