

# PARTIAL WORD AND EQUALITY PROBLEMS AND BANACH DENSITIES

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ABSTRACT. We investigate partial Equality and Word Problems for finitely generated groups. After introducing Upper Banach (UB) density on free groups, we prove that solvability of the Equality Problem on squares of UB-generic sets implies solvability of the whole Word Problem. In particular, we prove that solvability of generic EP implies WP. We then exploit another definition of generic EP, which turns out to be equivalent to generic WP. We characterize in different ways the class of groups with unsolvable UB-generic WP, proving that it contains that of infinite algorithmically finite groups, and it is contained in that of groups with unsolvable generic WP.

## 1. INTRODUCTION

One of the most striking results of the last century in group theory is certainly the existence of finitely presented groups with *unsolvable Word Problem (WP)*, independently proved in [3] and [28]. From a practical point of view, in computability and complexity theory it is often interesting to know the behavior of an algorithm on *almost all* inputs. A formalization of this approach, especially for the classical decision problems for groups, was given in [16]: the *generic version of a problem* is solvable if it is solvable on a *generic subset* of the input. A similar idea was already developed in group theory, essentially by Gromov [14], and was given a rigorous formulation by Arzhantseva and Olshanskii [1]. With this new generic approach, most of the known examples of unsolvable decision problems on groups turned out to be generically solvable, possibly even in linear time; see, for instance, [4, 5, 15, 16, 18, 19, 23]. This could be an issue, for example, for applications in group-based cryptography [25]. Remaining in context of the Word Problem, to the best of our knowledge, it is still unknown if there exists a finitely presented group with unsolvable generic WP. Various partial results have been obtained in this direction. In [22], computably presented, infinite, *algorithmically finite* groups (so-called *Dehn monsters*) were found. An algorithmically finite group is a group for which the Equality Problem is “extremely undecidable”: it is impossible to computably enumerate infinitely many pairwise distinct elements. It turns out that, with a suitable definition of the *partial Equality Problem (EP)*, infinite algorithmically finite groups can have solvable EP only on *negligible* sets. Moreover, the work [22] raised the question about the existence of finitely presented Dehn monsters, or at least of finitely presented groups whose EP is solvable only on non-generic sets. The first question is still open, other developments can be found in [20, 21]. For the latter question, the second author exhibited finitely presented groups with unsolvable *generic Equality Problem* [8].

The following is the first main result of this article, settled in the context of finitely generated groups, and it gives a more complete answer to the question raised in [22, Problem 1.5, b]. It is proved at the end of Section 2. We refer the reader to Section 1.1 for the relevant definitions.

**Theorem A.** *Let  $\Gamma$  be a finitely generated group. If there exists a finite set of generators  $X$  of  $\Gamma$  such that the Equality Problem is solvable on a set  $S \times S$ , where  $S \subset \mathbb{F}_X$  is generic (that is, the generic EP of  $\Gamma$  is solvable in the sense of [22]), then  $\Gamma$  has solvable Word Problem.*

In particular, no further assumptions are made on the group: this result holds for groups which are not necessarily amenable, computably presented (as it instead was in [8]). Note that, as a byproduct, this theorem shows for the first time that the generic EP in the sense of [22] does not depend on the choice of the finite set of generators. There is a simple idea behind this claim: up to left (or right) translations, a generic set contains all information about the whole Word Problem. We formalize this concept by introducing and studying *Banach densities* on free groups, densities that are, in a precise way, invariant under the action of an infinite sequence of translations. While the name of our densities refers to their classical analogues on  $\mathbb{Z}$ , the ideas leading to their definition and applications were partly inspired from the densities defined and studied by Solecki for any discrete group in [29].

They turn out to have other good invariance properties. For instance, the set of trivial words is negligible in a strong sense (cf. Theorem 2.5), which is a fundamental feature for investigations in genericity problems [12, 16]. We actually prove the following stronger version of Theorem A, via the definition of *Upper Banach generic (UB-generic) sets*; cf. Definition 2.1 and Theorem 2.9.

**Theorem B.** *Let  $\Gamma$  be a finitely generated group. If there exists a finite set of generators  $X$  of  $\Gamma$  such that the Equality Problem is solvable on a set  $S \times S$ , where  $S \subset \mathbb{F}_X$  is UB-generic, then  $\Gamma$  has solvable Word Problem.*

This suggests that these new densities might be interesting *per se*: we investigate the class of groups having solvable WP on UB-generic sets and characterize them as follows, cf. Theorem 4.1.

**Theorem C.** *Let  $\Gamma$  be a group generated by a finite set  $X$ . Then  $\Gamma$  has solvable UB-generic WP with respect to  $X$  if and only if there exists an infinite computably enumerable sequence of words in  $\mathbb{F}_X$  representing elements of strictly increasing length in  $\Gamma$ .*

A first consequence of this characterization is that solvability of UB-generic WP does not depend on the choice of the finite generating set. Moreover the class of groups whose WP is unsolvable on every UB-generic set can only contain torsion groups. Another straightforward consequence of Theorem C is the following, cf. Corollary 4.3.

**Corollary D.** *Let  $\Gamma$  be an infinite algorithmically finite group generated by a finite set  $X$ . The WP of  $\Gamma$  is unsolvable on every UB-generic set.*

Due to the exotic nature of algorithmically finite groups, we feel that their inclusion in a broader class of groups with nice and diverse characterizations can be helpful (cf. Theorem 4.1). Moreover, since UB-genericity is a weaker notion than

classic genericity, we prove that Dehn monsters also constitute the *first example of computably presented groups with unsolvable generic Word Problem*. This was obviously among the purposes of [22], but there the emphasis was on the Equality Problem. In light of our results, it seems appropriate to turn the attention to partial WP, or at least to consider a different definition for partial EP. In fact, proving Theorems A and B has required an analysis of the connection between Equality and Word Problem, which had sometimes been previously considered, but not deeply unraveled. This analysis revealed that the odd behavior exhibited in Theorem A is essentially due to the particular way of defining solvability of generic EP via *Fubini-genericity*: taking a more classical definition such as the one outlined in [16], we prove the expected equivalence between the two generic problems in Theorem 3.2.

**Theorem E.** *Let  $\Gamma$  be a group generated by a finite set  $X$ . Then  $\Gamma$  has solvable generic EP with respect to  $X$  in the sense of Definition 3.1 if and only if it has solvable generic WP with respect to  $X$ .*

Even if decidability of the generic EP is equivalent to that of the generic WP, we do not know if this is the case for the complexity: the partial relations obtained by our proofs are presented in Corollary 3.3 (see Remark 2.11 for the analogous discussion about Theorem A).

We devote the final part of this paper to taking a unifying look at these new and old classes of groups, defined according to the increasing level of (un)solvability of the partial WP, asking a few questions on the still unknown relations among them.

**1.1. Notation and preliminaries.** Throughout this paper,  $\Gamma$  is a finitely generated group. For a finite set of generators  $X$  we set  $|X| = d$ . We denote by  $\pi : \mathbb{F}_X \rightarrow \Gamma$  the canonical epimorphism from the free group on  $X$  to  $\Gamma$ . The normal subgroup  $\ker \pi \triangleleft \mathbb{F}_X$  of *trivial* words is often called the *Word Problem* of  $\Gamma$ . We denote by  $|g|_\Gamma$  the word length of  $g$  in  $\Gamma$  with respect to  $X$ , for  $\omega \in \mathbb{F}_X$  we simply write  $|\omega|$  instead of  $|\omega|_{\mathbb{F}_X}$ . Note that  $|g|_\Gamma = \min\{|\omega| : \omega \in \mathbb{F}_X, \pi(\omega) = g\}$ . For the  $k$ -th direct power  $\mathbb{F}_X^k$  of the free group  $\mathbb{F}_X$  we will consider the usual generators, so that  $|(\omega_1, \dots, \omega_k)|_{\mathbb{F}_X^k} = |\omega_1| + \dots + |\omega_k|$ . We denote with  $S_n(\Gamma)$  the sphere and with  $B_n(\Gamma)$  the ball of radius  $n$  in  $\Gamma$ , respectively. For the free group we simply write  $S_n$  instead of  $S_n(\mathbb{F}_X)$  and  $B_n$  instead of  $B_n(\mathbb{F}_X)$ . We let  $e$  denote the empty word in  $\mathbb{F}_X$ .

A set  $S \subset \mathbb{F}_X^k$  is *negligible* if

$$(1.1) \quad \lim_{n \rightarrow \infty} \frac{|S \cap B_n(\mathbb{F}_X^k)|}{|B_n(\mathbb{F}_X^k)|} = 0,$$

and *generic* if its complement in  $\mathbb{F}_X^k$  is negligible. The set  $S$  is *exponentially negligible* if it is negligible and the convergence in (1.1) is exponential, that is  $\beta^n \frac{|S \cap B_n(\mathbb{F}_X^k)|}{|B_n(\mathbb{F}_X^k)|} \rightarrow 0$ , for some  $\beta > 1$ . We say in this case that the complement is *exponentially generic* (see also [12, 16]). We will call *Fubini-generic* the special generic subsets of  $\mathbb{F}_X^2$  of the form  $S \times S \subset \mathbb{F}_X^2$ , with  $S$  generic in  $\mathbb{F}_X$ .

Setting  $\alpha := 2d - 1$ , an easy computation ensures that, when  $d > 1$ , there exist positive constants  $c_2, c_s, C_1, C_2$  such that

$$(1.2) \quad \begin{aligned} |S_n| &= c_s \alpha^n, & \alpha^n &\leq |B_n| \leq C_1 \alpha^n, \\ c_2(n+1)\alpha^n &\leq |B_n(\mathbb{F}_X^2)| &\leq C_2(n+1)\alpha^n. \end{aligned}$$

This implies, together with Cesaro-Stolz, that  $S \subset \mathbb{F}_X$  is (exponentially) negligible if and only if the sequence  $\frac{|S \cap S_n|}{|S_n|}$  (exponentially) tends to zero. This sequence is sometimes used in the literature to define, in an equivalent way, generic/negligible subsets of  $\mathbb{F}_X$ .

For definitions and basic facts on algorithms we refer to [10]. The group  $\Gamma$  has *solvable Word Problem (WP) on a subset*  $S \subset \mathbb{F}_X$  if there exists a partial algorithm that stops at least for every  $\omega \in S$ , and, if it stops, it establishes whether  $\omega$  is trivial or not. The group  $\Gamma$  has *solvable WP* with respect to  $X$  if it has solvable WP on  $\mathbb{F}_X$ ; it has *solvable generic WP* with respect to  $X$  if it has solvable WP on a generic subset  $S \subset \mathbb{F}_X$ . The group  $\Gamma$  has *solvable Equality Problem (EP) on a subset*  $T \subset \mathbb{F}_X^2$  if there exists a partial algorithm that stops at least for every  $(\omega_1, \omega_2) \in T$ , and establishes if  $\pi(\omega_1) = \pi(\omega_2)$ . The group  $\Gamma$  has *solvable Fubini-generic EP* with respect to  $X$  if it has solvable EP on a Fubini-generic subset of  $\mathbb{F}_X^2$ ; notice that this is exactly the definition of generic EP in the sense of [22].

All the previous generic problems can be stated in *exponentially generic* versions. Also, in all the previous problems we can replace *solvability* with *solvability in a (time) complexity class*  $C$ , see [16] for formal definitions and details. Notice that we will work with freely reduced words, that is words in  $\mathbb{F}_X$  instead of  $(X \cup X^{-1})^*$ . This is not relevant in our setting, since we assume, from now on, that complexity classes are defined by collections of complexity bounds which are also closed under addition of linear functions.

Not much is known about the invariance of generic solvability problems under change of the finite generating set  $X$  (see [16, Section 3]). Clearly solvability of the whole WP does not depend on  $X$ ; as a consequence, thanks to our Theorem A, solvability of Fubini-generic EP does not depend on the choice of the finite set of generators either.

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## 2. UPPER BANACH GENERIC WORD PROBLEM

We give our main definition concerning densities of subsets of free groups.

**Definition 2.1.** Let  $S$  be a subset of  $\mathbb{F}_X$ . We define the Lower and the Upper Banach densities ( $\underline{\mu}$  and  $\bar{\mu}$ , respectively) of  $S$  as:

$$\underline{\mu}(S) := \liminf_{n \rightarrow \infty} \min_{\omega \in \mathbb{F}_X} \frac{|S \cap \omega B_n|}{|B_n|}, \quad \bar{\mu}(S) := \limsup_{n \rightarrow \infty} \max_{\omega \in \mathbb{F}_X} \frac{|S \cap \omega B_n|}{|B_n|}.$$

A set  $S \subset \mathbb{F}_X$  is *Upper Banach generic (UB-generic)* for short if  $\bar{\mu}(S) = 1$ . A set  $N \subset \mathbb{F}_X$  is *UB-negligible* if its complement is UB-generic, that is if  $\underline{\mu}(N) = 0$ . Similarly, a set  $S \subset \mathbb{F}_X$  is *Lower Banach generic (LB-generic)* for short if

$\underline{\mu}(S) = 1$  and  $N \subset \mathbb{F}_X$  is *LB-negligible* if its complement is LB-generic, that is if  $\overline{\mu}(N) = 0$ .

The following proposition characterizes UB-generic subsets of free groups as those containing translates of any balls, and thus of any finite sets.

**Proposition 2.2.** *A subset  $S \subset \mathbb{F}_X$  is UB-generic if and only if for all  $n \in \mathbb{N}$  there exists  $\omega_n \in \mathbb{F}_X$  such that  $\omega_n B_n \subset S$ .*

*Proof.* The existence of a sequence  $\{\omega_n\}_{n \in \mathbb{N}} \subset \mathbb{F}_X$  such that  $\omega_n B_n \subset S$  for all  $n$  clearly implies the UB-genericity of  $S$ . For the converse, suppose  $S$  is UB-generic. We denote by  $N := S^c$  the complement of  $S$ , which is UB-negligible. Suppose by contradiction that there exists  $k \in \mathbb{N}$  such that  $\omega B_k \not\subset S$  (equivalently,  $\omega B_k \cap N \neq \emptyset$ ) for all  $\omega \in \mathbb{F}_X$ . One can check that (see, for instance, [8, Lemma 5.3]), for  $n$  big enough, the ball of radius  $n$  contains  $|S_{n-2k}|$  disjoint translates of  $B_k$ :

$$B_n \supset \bigsqcup_{i=1}^{|S_{n-2k}|} \omega_i B_k,$$

and then, for every  $\omega \in \mathbb{F}_X$  we have  $\omega B_n \supset \bigsqcup_i \omega \omega_i B_k$ . Since  $N$  contains at least a word for each translate of  $B_k$ , we have  $|N \cap \omega B_n| \geq |S_{n-2k}|$ , independently of  $\omega$ . Then if  $d > 1$ , by Equation (1.2)

$$\min_{\omega \in \mathbb{F}_X} \frac{|N \cap \omega B_n|}{|B_n|} \geq \frac{|S_{n-2k}|}{|B_n|} \geq \frac{c_s \alpha^{n-2k}}{C_1 \alpha^n} > 0,$$

that is impossible since  $\underline{\mu}(N) = 0$ . The case  $d = 1$  is actually a classical result (see [27, Lemma 1]), in our setting it is enough to notice that  $\omega B_n$  contains  $\lfloor \frac{n}{k} \rfloor \sim \frac{|B_n|}{|B_k|}$  disjoint translates of  $B_k$ .  $\square$

**Remark 2.3.** It follows from the above proposition that if  $S$  is UB-generic, then

$$S^{-1}S \supset (\bigcup \omega_n B_n)^{-1} (\bigcup \omega_n B_n) \supset \mathbb{F}_X.$$

It is clear from Definition 2.1 that UB-genericity is weaker than genericity, which is in turn weaker than LB-genericity. For a fixed non-trivial word  $\omega \in \mathbb{F}_X$  and some  $f : \mathbb{N} \rightarrow \mathbb{N}$ , define  $T_f := \bigcup_{n=1}^{\infty} \omega^{f(n)} B_n$  (analogous sets were considered, for instance, in [8, Remark 5.4]). The set  $T_f$  is always UB-generic but, choosing  $f$  growing fast enough, it is also negligible. Conversely, the set  $T_f^c$  is always non-LB-generic and, for the chosen  $f$ , it is also generic. Moreover, it is an easy exercise to define a set  $S$  such that  $S^{-1}S = \mathbb{F}_X$  and  $|S \cap S_n| \leq 1$  for all  $n$ . The last property ensures that such a set is not UB-generic. To summarize, the following hold.

$$S \text{ LB-generic} \xrightarrow{\not\Leftarrow} S \text{ generic} \xrightarrow{\not\Leftarrow} S \text{ UB-generic} \xrightarrow{\not\Leftarrow} S^{-1}S = \mathbb{F}_X.$$

**Definition 2.4.** Let  $\Gamma$  be a group generated by a finite set  $X$ . We say that  $\Gamma$  has *solvable UB-generic WP* with respect to  $X$  if it has solvable WP on an UB-generic subset of  $\mathbb{F}_X$ .

Note that our Theorem 4.1 implies, a posteriori, that this definition does not depend on the choice of the finite set of generators; cf. Corollary 4.2.

It is well known that if  $\Gamma$  is infinite, the set  $\ker \pi \subset \mathbb{F}_X$ , i.e. the set of the Word Problem, is negligible. For this reason, in order to study generic Word Problem, one can restrict the attention to the behavior of an algorithm on the non-trivial words. On the other hand, in the investigation of UB-generic WP, the negligibility

of  $\ker \pi$  is not enough, essentially because the intersection of an UB-generic set and a generic set can even be empty (e.g. the sets  $T_f$  and  $T_f^c$  in Remark 2.3). This is not the case for the intersection of an UB-generic set and a LB-generic set: one can easily check that this intersection is always UB-generic. The next theorem establishes that, if  $\Gamma$  is infinite, the set of trivial words is not only negligible, but also LB-negligible, thus ensuring that a set  $S$  is UB-generic if and only if  $S \setminus \ker \pi$  is UB-generic.

**Theorem 2.5.** *Let  $\Gamma$  be an infinite group generated by a finite set  $X$  and let  $\pi : \mathbb{F}_X \rightarrow \Gamma$  denote the canonical projection. Then  $\bar{\mu}(\ker \pi) = 0$ . Equivalently,  $\ker \pi$  is LB-negligible.*

*Proof.* If  $X$  consists of a single generator the claim is clear since  $\Gamma$  must be cyclic and  $\ker \pi$  is trivial; we assume  $d > 1$ . Let  $\omega \in \mathbb{F}_X$  with  $\pi(\omega) = g$ . Note that if  $\omega' \in \mathbb{F}_X$  is such that  $\pi(\omega') = g$ , then  $|\ker \pi \cap \omega S_n| = |\ker \pi \cap \omega' S_n|$ , and thus this quantity does not depend on the choice of representatives of  $g$ . Denoting by  $\gamma \in (\sqrt{2d-1}, 2d-1]$  the *cogrowth* of  $\Gamma$  (cf. [13]), the ratio  $\frac{|\ker \pi \cap \omega S_n|}{\gamma^n}$  tends to zero uniformly for  $g \in \Gamma$  (cf. [30, Theorem 2]). Therefore, we have  $\lim_{n \rightarrow \infty} \max_{\omega \in \mathbb{F}_X} \frac{|\ker \pi \cap \omega S_n|}{|S_n|} = 0$ , since

$$(2.1) \quad \frac{|\ker \pi \cap \omega S_n|}{|S_n|} = \frac{|\ker \pi \cap \omega S_n|}{\gamma^n} \frac{\gamma^n}{|S_n|}$$

and, being  $\gamma \leq 2d-1$ , the sequence  $\left\{ \frac{\gamma^n}{|S_n|} \right\}$  is uniformly bounded.

By Cesaro-Stoltz, also

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=0}^n \max_{\omega \in \mathbb{F}_X} |\ker \pi \cap \omega S_i|}{\sum_{i=0}^n |S_i|} = 0.$$

The result follows, as

$$\max_{\omega \in \mathbb{F}_X} \frac{|\ker \pi \cap \omega B_n|}{|B_n|} \leq \frac{\sum_{i=0}^n \max_{\omega \in \mathbb{F}_X} |\ker \pi \cap \omega S_i|}{\sum_{i=0}^n |S_i|}.$$

□

**Remark 2.6.** If  $\Gamma$  is infinite, by combining [30, Theorem 2], Equation (2.1) and the cogrowth criteria [9, 13], we get that  $\ker \pi$  is *exponentially LB-negligible* if and only if  $\Gamma$  is non-amenable. On the other hand, an UB-negligible set  $S$  is always *exponentially UB-negligible*, since in light of Proposition 2.2 the sequence  $\left\{ \min_{\omega \in \mathbb{F}_X} \frac{|S \cap \omega B_n|}{|B_n|} \right\}_{n \in \mathbb{N}}$  of Definition 2.1 is eventually 0.

For our comparisons between Equality and Word Problems, we need to switch between subsets of  $\mathbb{F}_X^2$  and of  $\mathbb{F}_X$ . To this purpose we define the following map

$$(2.2) \quad \tau : \mathbb{F}_X \times \mathbb{F}_X \rightarrow \mathbb{F}_X, (\omega_1, \omega_2) \mapsto \omega_1^{-1} \omega_2.$$

**Lemma 2.7.** *Let  $\Gamma$  be a group generated by a finite set  $X$ . Then  $\Gamma$  has solvable Equality Problem on a set  $T \subset \mathbb{F}_X \times \mathbb{F}_X$  if and only if  $\Gamma$  has solvable Word Problem on the set  $\tau(T) \subset \mathbb{F}_X$ .*

*Proof.* Let us denote with  $\mathcal{A}$  the algorithm solving the Equality Problem on  $T$ , we are going to describe an algorithm solving the Word Problem on the set  $\tau(T)$ . For every  $\omega, v \in \mathbb{F}_X$ , the word  $\omega$  is trivial if and only if  $\pi(v) = \pi(v\omega)$ . Let us denote by  $\{v_n\}_{n \in \mathbb{N}}$  a computable enumeration of  $\mathbb{F}_X$ . The algorithm takes  $\omega$  as an input

and runs the algorithm  $\mathcal{A}$  simultaneously on the pairs  $(v_n, v_n\omega)$ : if there exists  $n \in \mathbb{N}$  such that  $(v_n, v_n\omega) \in T$ , then the algorithm stops establishing if  $\omega$  is in  $\ker \pi$  or not. We conclude observing that  $\{\omega : \exists v \in \mathbb{F}_X : (v, v\omega) \in T\} = \tau(T)$ .

For the other direction, it is easy to see that solvability of WP on a set  $S \subset \mathbb{F}_X$  implies solvability of EP on all pairs of words having image through  $\tau$  in  $S$  itself, namely  $\tau^{-1}(S)$ . But if  $S = \tau(T)$  for some  $T \subset \mathbb{F}_X^2$  then  $T \subset \tau^{-1}(\tau(T))$ .  $\square$

For  $T \subset \mathbb{F}_X^2$ , if we denote with  $\overline{T} := \tau^{-1}(\tau(T))$ , solvability of EP on  $T$  implies solvability of EP on  $\overline{T}$ . For  $S \subset \mathbb{F}_X$ , if we also denote, by slight abuse of notation,  $\overline{S} := \{\alpha^{-1}\omega\alpha : \omega \in S, \alpha \in \mathbb{F}_X\} = S^{\mathbb{F}_X}$ , solvability of WP on  $S$  implies solvability of WP on  $\overline{S}$ . These facts can be also directly proved noticing that

$$\overline{T} \subset \{(\alpha\omega_1\beta, \alpha\omega_2\beta) : (\omega_1, \omega_2) \in T, \alpha, \beta \in \mathbb{F}_X\}.$$

Even if solvability of the EP on  $T$  is equivalent to that on  $\overline{T}$ , the complexity classes of these two problems are, a priori, different.

**Remark 2.8.** Let  $\Gamma$  be a group generated by a finite set  $X$ . If the WP on  $\tau(T) \subset \mathbb{F}_X$  is solvable in some complexity class  $C$  then the EP on  $T \subset \mathbb{F}_X^2$  is solvable in the complexity class  $C$ . It follows in fact from the proof of the previous lemma that the algorithm solving the EP on  $T$  is induced by the algorithm solving the WP on  $\tau(T)$  without altering the complexity. If the EP on  $T \subset \mathbb{F}_X^2$  is solvable in the complexity class  $C$  and  $T = \overline{T}$ , then WP on  $\tau(T) \subset \mathbb{F}_X$  is solvable in the complexity class  $C$ . Indeed, if  $T = \overline{T}$ , then  $(e, \omega) \in T$  for every  $\omega \in \tau(T)$  so the steps performed to establish if  $\omega$  is trivial are exactly those of  $\mathcal{A}$  on  $(e, \omega)$ .

**Theorem 2.9.** *Let  $\Gamma$  be a group generated by a finite set  $X$ . If  $\Gamma$  has solvable Equality Problem on a set  $S \times S$ , where  $S \subset \mathbb{F}_X$  is UB-generic, then  $\Gamma$  has solvable Word Problem.*

*Proof.* By virtue of Remark 2.3, if  $S$  is UB-generic then  $\tau(S \times S) = \mathbb{F}_X$ . By Lemma 2.7, the group  $\Gamma$  has solvable Word Problem.  $\square$

*Proof of Theorem A.* Suppose that  $\Gamma$  has Fubini-generic solvable EP, that is the EP is solvable on a set  $S \times S$  with  $S \subset \mathbb{F}_X$  generic. It follows from Remark 2.3 that the subset  $S$  is also UB-generic. By virtue of Theorem 2.9 the group  $\Gamma$  has solvable WP.  $\square$

As a consequence, while considering the Equality Problem on “small” square subsets  $S \times S \subset \mathbb{F}_X^2$  produces new concepts such as the algorithmic finiteness, considering its Fubini-generic solvability is simply equivalent to solvability of the classical WP, from the point of view of pure decidability. However, Fubini-genericity is still worth investigating in connection with complexity and with other decision problems, as we observe in the following remarks.

**Remark 2.10.** Note that an analogue of Theorem A for the Conjugacy Problem cannot exist. Indeed, in [4, 5] it is proved that, under suitable hypotheses on  $H$ , the Miller groups  $G(H)$  have solvable Conjugacy Problem on exponentially Fubini-generic (and in fact even bigger) sets. On the other hand, they have unsolvable Conjugacy Problem.

**Remark 2.11.** Even if solvability of the WP on  $\Gamma$  is equivalent to solvability of the EP on  $S \times S$  for any UB-generic set  $S \subset \mathbb{F}_X$ , their complexity classes are, a priori, different and possibly depend on the shape of  $S$ . Indeed, Proposition 2.2 only

ensures that for an UB-generic set  $S$  the function  $U_S(n) := \min\{|\omega| : \omega B_n \subset S\}$  is finite-valued but the natural upper bound for the complexity of our algorithm (described in the proof of Lemma 2.7) depends on the growth rate of  $U_S(n)$ .

One might still want to investigate the partial Equality Problem. A possible way to do it is to employ the natural definition of genericity in products (as already defined in [16]). As we show in the next section, considering the generic EP with this notion of genericity does not lead to new behavior either.

### 3. GENERIC WORD PROBLEM AND GENERIC EQUALITY PROBLEM

**Definition 3.1.** We say that a group  $\Gamma$  generated by a finite set  $X$  has *solvable generic EP* with respect to  $X$ , if it has solvable EP on a generic set of  $\mathbb{F}_X^2$  (in the sense of Equation (1.1)).

**Theorem 3.2.** *Let  $\Gamma$  be a group generated by a finite set  $X$ . Then  $\Gamma$  has solvable generic Word Problem with respect to  $X$  if and only if it has solvable generic Equality Problem with respect to  $X$ . Moreover, the Word Problem on  $\Gamma$  is solvable only on negligible subsets of  $\mathbb{F}_X$  if and only if the Equality Problem on  $\Gamma$  is solvable only on negligible subsets of  $\mathbb{F}_X^2$ .*

An easy consequence of the previous theorem and Remark 2.8 is the following.

**Corollary 3.3.** *Let  $\Gamma$  be a group generated by a finite set  $X$ . If the generic WP with respect to  $X$  is solvable in some complexity class  $C$  then the generic EP with respect to  $X$  is solvable in the complexity class  $C$ . Vice versa, if the EP on a generic set  $T \subset \mathbb{F}_X^2$ , with  $T = \bar{T}$ , is solvable in some complexity class  $C$ , then the generic WP with respect to  $X$  is solvable in the complexity class  $C$ .*

To prove Theorem 3.2 we will need two lemmas to compare densities of subsets of  $\mathbb{F}_X$  and of  $\mathbb{F}_X \times \mathbb{F}_X$ . Recall that  $|X| = d$  and  $\alpha = 2d - 1$ , and recall the definition of the map  $\tau : \mathbb{F}_X \times \mathbb{F}_X \rightarrow \mathbb{F}_X$  in Equation (2.2). For  $s \in \mathbb{F}_X$  and  $n \in \mathbb{N}$ , we define the subset

$$P(s, n) := \{\omega \in \mathbb{F}_X : |\omega| + |\omega s| \leq n\},$$

which we will need to estimate the density of preimages under  $\tau$ .

**Lemma 3.4.** *For every reduced word  $s = s_k s_{k-1} \cdots s_1 \in \mathbb{F}_X$ ,  $s_i \in X \cup X^{-1}$ , with  $|X| > 1$ , we have*

$$P(s^{-1}, n) = \bigcup_{i=0}^k B_{\frac{n-k}{2}} s_i \cdots s_1.$$

*Proof.* Informally, we want to show that in the left Cayley graph of  $\mathbb{F}_X$  the set  $P(s^{-1}, n)$  is the “neighborhood of radius  $\frac{n-k}{2}$ ” of the geodesic from the empty word to  $s$ .

Let  $\omega \in P(s^{-1}, n)$ . Suppose that the product of  $\omega$  with  $s^{-1} = s_1^{-1} \cdots s_k^{-1}$  induces exactly  $i$  cancellations: this implies that  $\omega = v s_i \cdots s_1$  for some  $v$  not ending in  $s_i^{-1}$  or  $s_{i+1}$  and  $|\omega s^{-1}| = |v s_{i+1}^{-1} \cdots s_k^{-1}| = |v| + k - i$ . Since  $\omega \in P(s^{-1}, n)$  we have

$$n \geq |\omega| + |\omega s^{-1}| = |v| + i + |v| + k - i,$$

that is  $|v| \leq \frac{n-k}{2}$  and therefore  $P(s^{-1}, n) \subset \bigcup_{i=0}^k B_{\frac{n-k}{2}} s_i \cdots s_1$ .



To prove the other inclusion, let now  $\omega \in \bigcup_{i=0}^k B_{\frac{n-k}{2}} s_i \cdots s_1$ . This means that there exist  $i$  with  $0 \leq i \leq k$  and  $v \in B_{\frac{n-k}{2}}$  such that  $\omega = v s_i \cdots s_1$ . Then

$$|\omega| + |\omega s^{-1}| = |v s_i \cdots s_1| + |v s_{i+1}^{-1} \cdots s_k^{-1}| \leq 2 \left( \frac{n-k}{2} \right) + i + k - i \leq n,$$

that is  $\omega \in P(s^{-1}, n)$ .  $\square$

**Lemma 3.5.** *For any  $S \subset \mathbb{F}_X$ , for any  $T \subset \mathbb{F}_X^2$ , the following hold:*

- (1)  *$S$  is (exponentially) negligible if and only if  $\tau^{-1}(S)$  is (exponentially) negligible;*
- (2)  *$S$  is (exponentially) generic if and only if  $\tau^{-1}(S)$  is (exponentially) generic;*
- (3) *if  $T$  is (exponentially) generic then  $\tau(T)$  is (exponentially) generic;*
- (4) *if  $\tau(T)$  is (exponentially) negligible then  $T$  is (exponentially) negligible.*

*Proof.* When  $d = 1$  all claims follow from simple computations in the lattice  $\mathbb{Z}^2$ .

Suppose  $d > 1$ . Observe that

$$(3.1) \quad |\tau^{-1}(S) \cap B_n(\mathbb{F}_X^2)| = \sum_{s \in S} |\{(\omega_1, \omega_2) : |\omega_1| + |\omega_2| \leq n, \omega_1^{-1} \omega_2 = s\}| \\ = \sum_{s \in S} |P(s, n)|.$$

Notice that, if  $|s| > n$ , the set  $P(s, n)$  is empty. We are going to prove that if  $S$  is (exponentially) negligible, then  $\tau^{-1}(S)$  is (exponentially) negligible.

By virtue of Equation (1.2) and Lemma 3.4, we have, for any  $s \in S_k$ ,  $|P(s, n)| \leq (k+1) |B_{(n-k)/2}| \leq C_1 (k+1) \alpha^{(n-k)/2}$ . As a consequence

$$\frac{|\tau^{-1}(S) \cap B_n(\mathbb{F}_X^2)|}{|B_n(\mathbb{F}_X^2)|} \leq \frac{\sum_{s \in S} |P(s, n)|}{c_2 (n+1) \alpha^n} \leq \frac{C_1}{c_2} \sum_{k=0}^n \frac{|S \cap S_k|}{\alpha^{\frac{n+k}{2}}}.$$

Assume that  $\beta \geq 1$  is such that  $\beta^n \frac{|S \cap B_n|}{|B_n|} \rightarrow 0$ . Note that this is equivalent to  $S$  being negligible when  $\beta = 1$ , and  $S$  being exponentially negligible when  $\beta > 1$ . Without loss of generality we can also assume  $\beta < \sqrt{\alpha}$ . By Equation (1.2), we have that  $\beta^n \frac{|S \cap S_n|}{\alpha^n} \leq \beta^n \frac{|S \cap B_n|}{\alpha^n} \rightarrow 0$ .

In particular, for every  $\varepsilon > 0$ , there exists  $\bar{k}$  such that  $\beta^k \frac{|S \cap S_k|}{\alpha^k} < \varepsilon$  for every  $k > \bar{k}$ . Moreover, for every fixed  $\bar{k}$  there exists  $\bar{n} \in \mathbb{N}$  such that  $\bar{n} > \bar{k}$  and  $\left(\frac{\beta}{\sqrt{\alpha}}\right)^{\bar{n}} \sum_{k=0}^{\bar{k}} \frac{|S \cap S_k|}{\alpha^{k/2}} < \varepsilon$ . For any  $n > \bar{n}$  we have

$$\beta^n \frac{|\tau^{-1}(S) \cap B_n(\mathbb{F}_X^2)|}{|B_n(\mathbb{F}_X^2)|} \leq \frac{C_1}{c_2} \left( \frac{\beta^n}{\alpha^{n/2}} \sum_{k=0}^{\bar{k}} \frac{|S \cap S_k|}{\alpha^{k/2}} + \sum_{k=\bar{k}+1}^n \beta^k \frac{|S \cap S_k|}{\alpha^k} \left(\frac{\beta}{\sqrt{\alpha}}\right)^{n-k} \right) \\ \leq \frac{C_1}{c_2} \left( \left(\frac{\beta}{\sqrt{\alpha}}\right)^{\bar{n}} \sum_{k=0}^{\bar{k}} \frac{|S \cap S_k|}{\alpha^{k/2}} + \varepsilon \sum_{k=\bar{k}+1}^n \left(\frac{\beta}{\sqrt{\alpha}}\right)^{n-k} \right) \\ \leq \frac{C_1}{c_2} \left( \varepsilon + \varepsilon \sum_{t=0}^{\infty} \left(\frac{\beta}{\sqrt{\alpha}}\right)^t \right) \leq \varepsilon \frac{C_1}{c_2} \left( 1 + \frac{\sqrt{\alpha}}{\sqrt{\alpha} - \beta} \right).$$

Therefore  $\tau^{-1}(S)$  is negligible, and if  $\beta > 1$  even exponentially negligible.

Now suppose that  $S$  is not (exponentially) negligible. By Lemma 3.4 we have that  $|P(s, n)| = n + 1$  for  $s \in S_n$ . Combining with Equation (3.1) and Equation (1.2) we have

$$\frac{|\tau^{-1}(S) \cap B_n(\mathbb{F}_X^2)|}{|B_n(\mathbb{F}_X^2)|} \geq \frac{\sum_{s \in S_n} |P(s, n)|}{C_2(n+1)\alpha^n} \geq \frac{1}{C_2} \frac{|S \cap S_n|}{\alpha^n}$$

proving that  $\tau^{-1}(S)$  is not (exponentially) negligible. This completes the proof of (1).

The second claim easily follows from (1) by taking complements and observing that  $\tau^{-1}(S^c) = (\tau^{-1}(S))^c$ . Finally, (3) and (4) easily follow by taking  $S = \tau(T)$  and observing that  $T \subset \overline{T} = \tau^{-1}(\tau(T))$ .  $\square$

*Proof of Theorem 3.2.* Suppose  $\Gamma$  has solvable WP on  $S \subset \mathbb{F}_X$  generic. This means, by virtue of Lemma 2.7, that  $\Gamma$  has solvable EP on  $\tau^{-1}(S)$ , which by Lemma 3.5 is generic. For the converse, suppose  $\Gamma$  has solvable EP on a generic set  $T \subset \mathbb{F}_X^2$ . By virtue of Lemma 2.7, the group  $\Gamma$  has solvable WP on  $\tau(T)$ , which by Lemma 3.5 is generic.

Suppose that solvability of the WP on  $S \subset \mathbb{F}_X$  implies that  $S$  is negligible. Assume that  $\Gamma$  has solvable EP on  $T$  with  $T$  non-negligible. Then, by Lemma 3.5 the subset  $\tau(T)$  is non-negligible and by Lemma 2.7 the group  $\Gamma$  has solvable WP on  $\tau(T)$ , that is a contradiction. An analogous argument proves that if EP is solvable only on negligible sets, then the same is true for the WP.  $\square$

**Remark 3.6.** Clearly, Theorem 3.2 holds true replacing generic (resp. negligible) with exponentially generic (resp. exponentially negligible).

In [12] (and elsewhere), the length in  $\mathbb{F}_X \times \mathbb{F}_X$  used to define genericity is  $\ell(\omega_1, \omega_2) := \max\{|\omega_1|, |\omega_2|\}$ . The ball of radius  $n$  in  $\mathbb{F}_X^2$  with respect to this length is  $B_n \times B_n$ . An analogue of Lemma 3.5 (and therefore of Theorem 3.2) can be proved in this setting by using similar arguments. The role of the set  $P(s, n)$  is played, in that case, by the set  $B_n \cap B_n s$ . One can indeed show, similarly to Lemma 3.4, that  $B_n \cap B_n s$  is the ball of radius  $n - \frac{|s|}{2}$  centered in the middle point of the geodesic from the empty word to  $s$ . Moreover, this analogue of Lemma 3.5 gives a generalization of [12, Lemma 3.2] which holds in non-exponential setting.

#### 4. UPPER BANACH GENERIC WORD PROBLEM AND ALGORITHMIC FINITENESS

A group  $\Gamma$  generated by a finite set  $X$  is algorithmically finite if there does not exist a computable enumeration of an infinite set of words in  $\mathbb{F}_X$  projecting onto pairwise distinct elements of  $\Gamma$ , or, equivalently if (cf. [22]):

- solvability of EP on  $S \times S$  implies that  $\pi(S)$  is finite;
- for any infinite computably enumerable set  $S \subset \mathbb{F}_X$  we have that  $S^{-1}S \cap \ker \pi \neq e$ ;
- solvability of WP on  $S^{-1}S$  implies that  $\pi(S)$  is finite (Lemma 2.7).

Note that algorithmic finiteness does not depend on the choice of the finite generating set, cf. [22, Lemma 2.4].

We now characterize, in a similar fashion, solvability of the UB-generic WP (see Definition 2.4), and therefore groups without this property.

**Theorem 4.1.** *Let  $\Gamma$  be an infinite group generated by a finite set  $X$ . The following are equivalent:*

- (1)  $\Gamma$  has solvable UB-generic WP with respect to  $X$ ;
- (2) there exists  $S \subset \mathbb{F}_X$  computably enumerable, UB-generic and such that  $S \cap \ker \pi = \emptyset$ ;
- (3) there exists a computably enumerable sequence  $\{\omega_n\}_{n \in \mathbb{N}} \subset \mathbb{F}_X$  such that  $|\omega_n|_\Gamma > n$  for all  $n \in \mathbb{N}$ ;
- (4) there exists a computably enumerable sequence  $\{\omega_n\}_{n \in \mathbb{N}} \subset \mathbb{F}_X$  such that  $|\omega_{n+1}|_\Gamma > |\omega_n|_\Gamma$  for all  $n \in \mathbb{N}$ .

*Proof.*

(1)  $\iff$  (2)

Suppose  $\Gamma$  has solvable UB-generic WP, then there exists an algorithm solving the Word Problem on an UB-generic subset of inputs  $S' \subset \mathbb{F}_X$ . The subset  $S := S' \setminus \ker \pi$  is computably enumerable and, by virtue of Theorem 2.5, UB-generic. Vice versa, the computable enumeration of  $S$  solves the WP on the UB-generic set  $S$ .

(2)  $\iff$  (3)

Suppose that  $S$  is a computably enumerable UB-generic subset of non-trivial words. By an elementary argument, the subset  $\Omega := \{(n, \omega) : \omega B_n \subset S\} \subset \mathbb{N} \times \mathbb{F}_X$  is also computably enumerable. By virtue of Proposition 2.2, the set  $\Omega_n := \{\omega : (n, \omega) \in \Omega\}$  is non-empty for all  $n \in \mathbb{N}$ . Let us denote by  $\omega_n$  the first element of  $\Omega_n$  in the computable enumeration of  $\Omega$ . Clearly  $\{\omega_n\}_{n \in \mathbb{N}}$  is computably enumerable. Finally,  $\omega_n B_n \cap \ker \pi \subset S \cap \ker \pi = \emptyset$  implies that  $|\omega_n|_\Gamma > n$ . Conversely,  $S := \bigcup_{n \in \mathbb{N}} \omega_n B_n$  is computably enumerable, UB-generic and such that  $S \cap \ker \pi = \emptyset$ .

(3)  $\iff$  (4)

Suppose that  $\{\omega_n\}_{n \in \mathbb{N}}$  is a computably enumerable sequence of  $\mathbb{F}_X$  such that  $|\omega_n|_\Gamma > n$  for all  $n \in \mathbb{N}$ . We define inductively a subsequence  $\{\omega_{k_n}\}_{n \in \mathbb{N}}$  as follows:  $k_1 := 1$  and  $k_{n+1} := |\omega_{k_n}|$  for all  $n \geq 1$ . The new sequence is still computably enumerable and with the property that, for every  $n \in \mathbb{N}$ ,

$$k_n < |\omega_{k_n}|_\Gamma \leq k_{n+1},$$

and thus  $|\omega_{k_{n+1}}|_\Gamma > |\omega_{k_n}|_\Gamma$  for all  $n \in \mathbb{N}$ . For the reverse implication notice that if the sequence of non-negative integers  $\{|\omega_n|_\Gamma\}_{n \in \mathbb{N}}$  is strictly increasing, then  $\{|\omega_{n+2}|_\Gamma\}_{n \in \mathbb{N}}$  is superlinear. Therefore  $\{\omega_{n+2}\}_{n \in \mathbb{N}} \subset \mathbb{F}_X$  is a sequence with the desired property.  $\square$

The equivalence of (1) and (3) (and (4)), yields the following.

**Corollary 4.2.** *If  $\Gamma$  has solvable UB-generic WP with respect to a finite generating set  $X$  then it has solvable UB-generic WP with respect to any finite generating set.*

This shows that using the Upper Banach density to measure the Word Problem is a natural choice. In order to study intrinsic properties of  $\Gamma$  it is maybe even more natural than the classical density. In particular, the relation with the length in  $\Gamma$  allows us to state the following corollary.

**Corollary 4.3.** *Let  $\Gamma$  be a infinite, finitely generated group. If  $\Gamma$  is algorithmically finite then it has unsolvable UB-generic Word Problem and therefore unsolvable generic Word Problem with respect to any finite set of generators.*

*Proof.* The existence of the sequence in (4) of Theorem 4.1 contradicts the definition of algorithmic finiteness.  $\square$

Let us denote with  $\mathcal{C}_1, \mathcal{C}_2$  and  $\mathcal{C}_3$  the classes of infinite finitely generated groups defined by the following properties:

$$\begin{aligned}\mathcal{C}_1 &:= \{\Gamma : \text{WP on } S \implies |\pi(S)| < \infty\}, \\ \mathcal{C}_2 &:= \{\Gamma : \text{WP on } S^{-1}S \implies |\pi(S)| < \infty\}, \\ \mathcal{C}_3 &:= \{\Gamma : \text{WP on } S \implies S \text{ is not UB-generic}\}.\end{aligned}$$

Equivalently,

$$\begin{aligned}\mathcal{C}_1 &= \{\Gamma : S \text{ computably enumerable, } |\pi(S)| = \infty \implies S \cap \ker \pi \neq \emptyset\}, \\ \mathcal{C}_2 &= \{\Gamma : S \text{ computably enumerable, } |S| = \infty \implies S^{-1}S \cap \ker \pi \neq e\}, \\ \mathcal{C}_3 &= \{\Gamma : S \text{ computably enumerable, UB-generic} \implies S \cap \ker \pi \neq \emptyset\}.\end{aligned}$$

Note that the properties defining the classes  $\mathcal{C}_1, \mathcal{C}_2$  and  $\mathcal{C}_3$ , hold equivalently for one or for all finite sets of generators, which we omit in the definition. We let

$$\mathcal{C}_4 := \{\Gamma : \text{WP on } S \subset \mathbb{F}_X \implies S \text{ is not generic, for all finite generating sets } X\},$$

equivalently,

$$\mathcal{C}_4 = \{\Gamma : S \subset \mathbb{F}_X \text{ c. e., generic} \implies S \cap \ker \pi \neq \emptyset \text{ for all finite generating sets } X\}.$$

Finally let us denote with  $\mathcal{C}_5$  the class of infinite finitely generated groups with unsolvable word problem.

The below chain of inclusions swiftly follows from our investigation

$$(4.1) \quad \mathcal{C}_1 \subset \mathcal{C}_2 \subset \mathcal{C}_3 \subset \mathcal{C}_4 \subset \mathcal{C}_5.$$

Since we can always assume that a finitely generated group  $\Gamma$  has solvable WP on a finite set  $S$ , the WP is solvable on its conjugacy closure  $\bar{S}$ . This easily implies that all the conjugacy classes of a group in  $\mathcal{C}_1$  are finite. It was already observed in [22] that algorithmically finite groups (our class  $\mathcal{C}_2$ ) must be periodic. This is also the case for groups with unsolvable UB-generic WP (our class  $\mathcal{C}_3$ ): in fact, an element of infinite order provides a sequence like in (4) of Theorem 4.1. The computably presented groups in  $\mathcal{C}_5$  have infinitely many conjugacy classes, as stated in [24, Corollary 1]. Indeed, assuming that  $\ker \pi$  is computably enumerable and that there exists a finite set  $S \subset \mathbb{F}_X$  containing an element of each non-trivial conjugacy class, it is easy to see that  $\mathbb{F}_X \setminus \ker \pi = \bar{S} \cdot \ker \pi$  and then also the set of non-trivial words is computably enumerable.

To our best knowledge, the only strict inclusion in (4.1) is that of  $\mathcal{C}_4$  in  $\mathcal{C}_5$ . Moreover, the only examples of computably presented groups in  $\mathcal{C}_4$  actually live in  $\mathcal{C}_2$ . This inspires the following questions.

**Question 4.4.** *Are any of the inclusions in (4.1) strict and/or trivial? Or one could ask the same question within various subclasses of computably presented groups, such as*

- residually finite (see also [22, Problem 3.3] and [20, 21]);
- amenable (see also [22, Problem 3.4] and [12, Theorem 2.3]); with computable Følner sets [7]; of intermediate growth;
- sofic; with subrecursive sofic dimension (see [6]).

If  $\Gamma$  is amenable there exist notions of Banach densities with respect to a Følner sequence (see, for instance, [11]), which are generalizations of the classical Banach density for  $\mathbb{Z}$ , cf. [27]. Once more, these densities are closely related to those considered in [29] where, moreover, it is proved that they exhibit peculiar behavior on amenable groups; cf. also [2]. These notions allow to formulate the extension of Erdős Sumset conjecture to amenable groups: if  $A \subset \Gamma$  has positive UB-density (with respect to a Følner sequence) then there exist two infinite subsets  $B, C \subset \Gamma$  such that  $BC \subset A$  (this conjecture was recently proved in [26]).

**Question 4.5.** *Are the Banach densities of  $A$  in  $\Gamma$  and  $\pi^{-1}(A)$  in  $\mathbb{F}_X$  related?*

Analogous investigations were carried out in [17] for free abelian groups. A positive answer to this question could lead to intriguing implications and new questions in relation with Question 4.4 and the aforementioned conjecture–now theorem.

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